Noncommutative Gröbner bases and automated proofs of operator statements

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Automated proofs of operator statements

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DISCRETE MATHEMATICS AND ITS APPLICATIONS
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HANDBOOK OF LINEAR ALGEBRA

SECOND EDITION

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Noncommutative polynomials

$$\text{Noncommutative polynomials} \quad = \quad \text{elements in free algebra } R\langle X\rangle$$

$$= \quad \sum_{i=1}^{d} c_i \cdot \quad x_{i,1} \ldots x_{i,k_i}$$

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finite words over X

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finite words over X

Multiplication $=$ Concatenation of words

$$(x_1 \dots x_k) \cdot (x'_1 \dots x'_l) = x_1 \dots x_k x'_1 \dots x'_l$$

Example: $(ab - 1) \cdot (ba + 1) = abba + ab - ba - 1$

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Two-sided ideals For $f_1, \dots, f_r \in R\langle X\rangle$

$$(f_1, \dots, f_r) = \left\{ \sum_{i,j} a_{i,j} \cdot f_i \cdot b_{i,j} \;\middle|\; a_{i,j}, b_{i,j} \in R\langle X\rangle \right\}$$

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Fact Ideal membership problem $f \overset{?}{\in} (f_1, \dots, f_r)$ is semi-decidable
(e.g., using Gröbner bases)

Operator statements

Operators

- $0, A, B, C, \ldots$
- $S + T, \; S \cdot T, \; f(T_1, \ldots, T_n)$

Linearity = abelian (partial) addition + assoc. (partial) mult. + dist.

Operator statements

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$^*,\ \cdot^{\mathsf{T}},\ \|\cdot\|,\ \otimes, \ldots$

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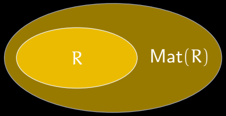

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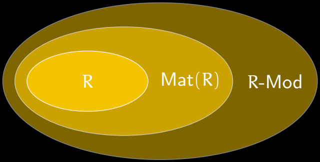

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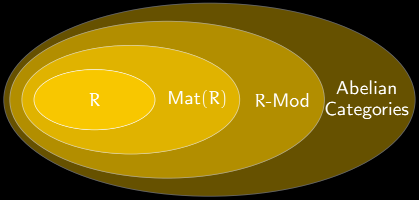

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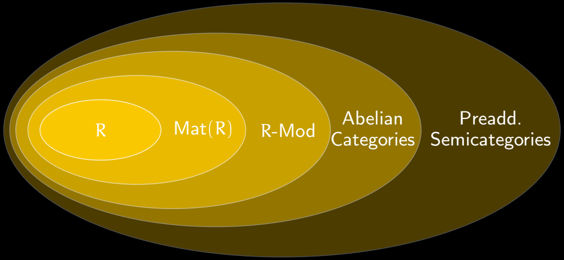

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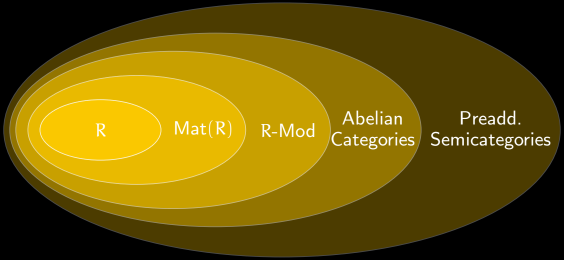
$S = T, \quad \neg\, \varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad (\varphi \Rightarrow \psi), \quad \exists X : \varphi, \quad \forall X : \varphi$



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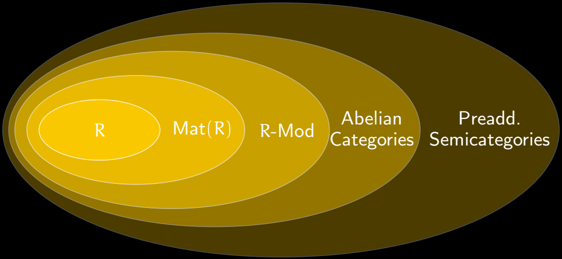
Definition An operator statement is universally true if it follows from linearity



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- Fact: Determining universal truth is not decidable
  $\Rightarrow$ Algorithm that terminates on all inputs cannot exist

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Definition An operator statement is universally true if it follows from linearity

- Fact: Determining universal truth is not decidable
  $\Rightarrow$ Algorithm that terminates on all inputs cannot exist
- Best we can hope for: (effective) semi-decision procedure
  $\rightarrow$ Can be obtained using computer algebra

"The Moore-Penrose inverse is unique"

Recall: B is Moore-Penrose inverse of A if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

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From identities to polynomials

$$L = R \iff L - R = 0$$

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$$L = R \qquad \Longleftrightarrow \qquad l - r \; \in \; \mathbb{Z}\langle \mathbf{X} \rangle$$

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$$L = R \iff l - r \in \mathbb{Z}\langle \mathbf{X} \rangle$$

$$B = \ldots = C \iff b - c \in (f_1, \ldots, f_{12})$$

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Theorem (Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$$\forall \mathbf{X} : \bigwedge_{i=1}^{m} P_i = Q_i \Rightarrow S = T \quad \text{iff} \quad s - t \in (p_1 - q_1, \ldots, p_m - q_m)$$

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Theorem (Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$\forall \mathbf{X} : \bigwedge_{i=1}^{m} P_i = Q_i \Rightarrow S = T$ iff

$$s - t = \sum_{i,j} a_{i,j} \cdot (p_i - q_i) \cdot b_{i,j}$$

- *"cofactor representation"*
- computable with Gröbner bases

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Proof Using our software package `operator_gb`...

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Proof Using our software package operator_gb...

```
sage: from operator_gb import *
sage: assumptions = [a*b*a - a,...]
sage: certify(assumptions, b - c)
```

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b - c = (-c + c*a*c) + b*c_adj*(-a_adj + a_adj*b_adj*a_adj)
        - b*a*c*(-a*b + b_adj*a_adj) - b*(-a + a*c*a)*b
        + b*(-a*c + c_adj*a_adj) - b*(-a*c + c_adj*a_adj)*b_adj*a_adj
        - (-b + b*a*b) + (-c*a + a_adj*c_adj)*b*a*c
        - (-a_adj + a_adj*c_adj*a_adj)*b_adj*c + c*(-a + a*b*a)*c
        - (-b*a + a_adj*b_adj)*c + a_adj*c_adj*(-b*a + a_adj*b_adj)*c
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b - c = (-c + c*a*c) + b*c_adj*(-a_adj + a_adj*b_adj*a_adj)
        - b*a*c*(-a*b + b_adj*a_adj) - b*(-a + a*c*a)*b
        + b*(-a*c + c_adj*a_adj) - b*(-a*c + c_adj*a_adj)*b_adj*a_adj
        - (-b + b*a*b) + (-c*a + a_adj*c_adj)*b*a*c
        - (-a_adj + a_adj*c_adj*a_adj)*b_adj*c + c*(-a + a*b*a)*c
        - (-b*a + a_adj*b_adj)*c + a_adj*c_adj*(-b*a + a_adj*b_adj)*c
```

- Software produces cofactor representation (= algebraic proof)

$\Rightarrow$ Operator statement is universally true

Determining universal truth

Quasi-identities

(Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$$\forall \mathbf{X} : \bigwedge_{i=1}^{m} P_i = Q_i \;\Rightarrow\; S = T \qquad \text{iff} \qquad s - t \in (p_1 - q_1, \ldots, p_m - q_m)$$

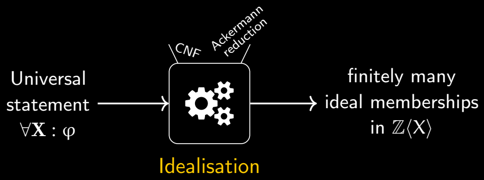
Determining universal truth

Universal statements

Universal
statement
$\forall \mathbf{X} : \varphi$

Determining universal truth

Universal statements



Determining universal truth

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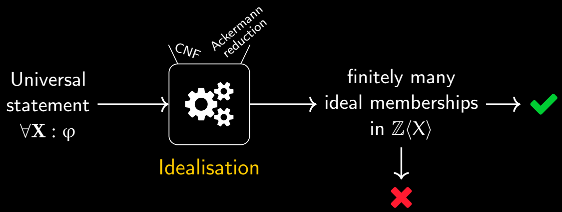


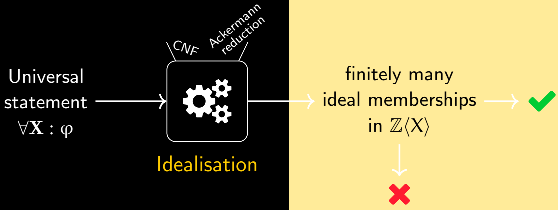
Theorem (H., Raab, Regensburger '22b)
A universal statement is universally true iff its idealisation is true

H., Raab, Regensburger. *Universal truth of operator statements via ideal membership*. preprint. 2022.

Determining universal truth

Universal statements

efficiently verifiable using Gröbner bases



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H., Raab, Regensburger. *Universal truth of operator statements via ideal membership*. preprint. 2022.

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \dots, 1/\sigma_r, 0, \dots, 0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X \in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim\limits_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim\limits_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum\limits_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum\limits_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim\limits_{E\to 0}(A + E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

“Every matrix has a Moore-Penrose inverse”

“*Every matrix* *has a Moore-Penrose inverse*”

"Every matrix has a Moore-Penrose inverse"

Fact: A matrix $\quad\Rightarrow\quad \exists\, P, Q \;:\; PA^*A = A$ and $AA^*Q = A$

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Claim $\exists X : (PA^*A = A \wedge AA^*Q = A) \;\Rightarrow\; \mathrm{mp}(A, X)$

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Strategy

1. Derive explicit expression for X
2. Reformulate statement as a universal statement
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```
sage: assumptions =  [a - p*a_adj*a,...]
sage: I = NCIdeal(assumptions + pinv(a,x))
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```
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```

$\Rightarrow$ $X = A^*QP^*$ is MP-inverse of A

(can be certified using the software)

Existential statements

In the previous example, we found a suitable polynomial expression.

Question Was this just luck?

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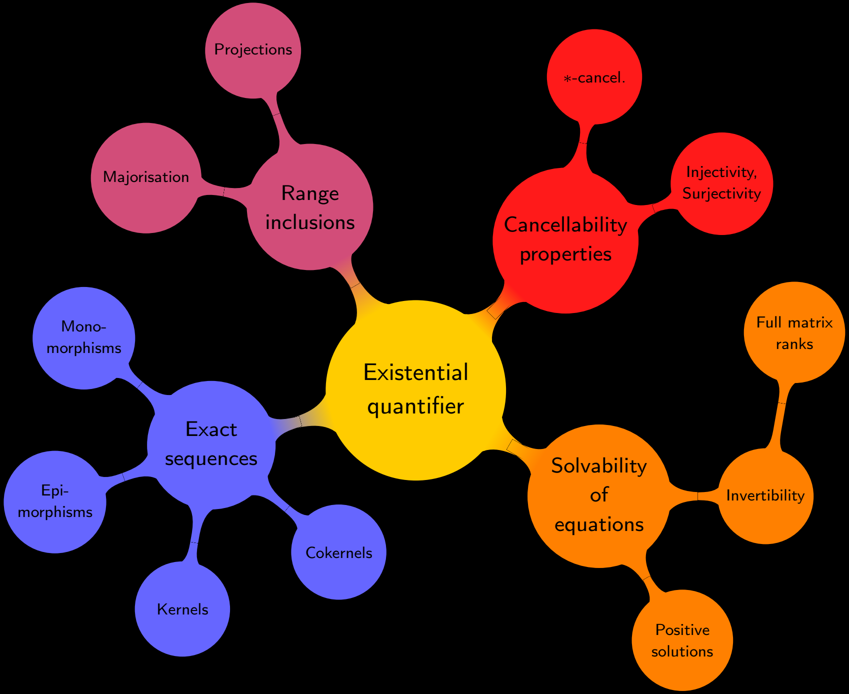
Such expressions always exist and the possible candidates are enumerable.

- Enumerating all possible expressions is hopeless
- Requires good heuristics $\longrightarrow$ provided by computer algebra
  (H., Raab, Regensburger '22a)
- Several heuristics implemented in `operator_gb`
  (ansatz, ideal/subalgebra intersections, hom. part, monomial part,...)

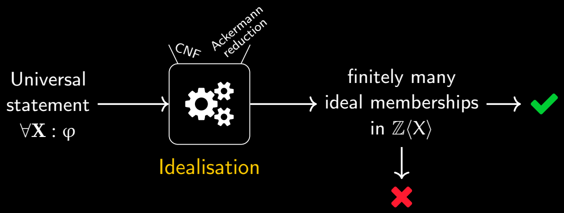
H., Raab, Regensburger. *Computing Elements of Certain Form in Ideals to Prove Properties of Operators*. In: Mathematics in Computer Science. 2022.

Existential
quantifier


Existential quantifier
Range inclusions
Projections
Majorisation
Cancellability properties
∗-cancel.
Injectivity, Surjectivity
Solvability of equations
Invertibility
Full matrix ranks
Positive solutions
Exact sequences
Mono-morphisms
Epi-morphisms
Kernels
Cokernels

Determining universal truth

Universal statements



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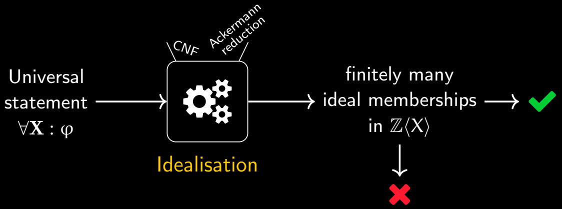
Determining universal truth

Universal statements



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To treat all operator statements $\rightsquigarrow$ combine with Herbrand's theorem
+ Heuristics

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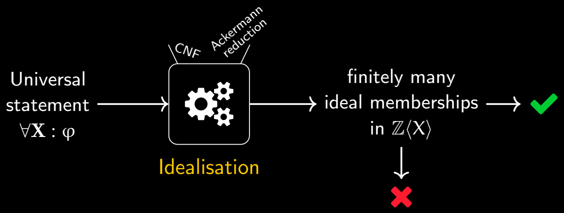
Determining universal truth

General operator statements



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Determining universal truth

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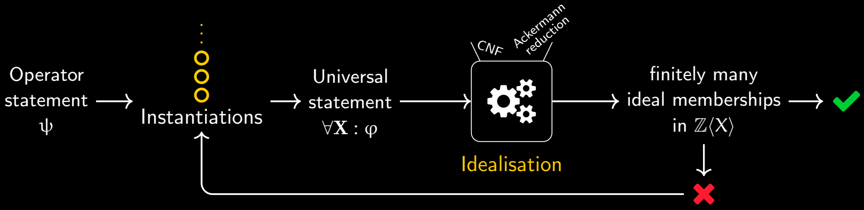

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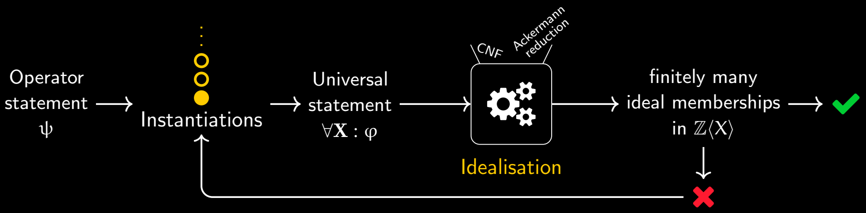

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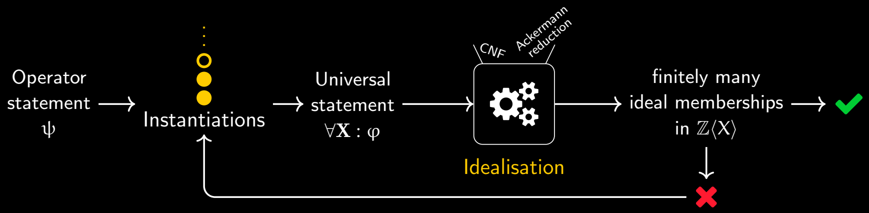

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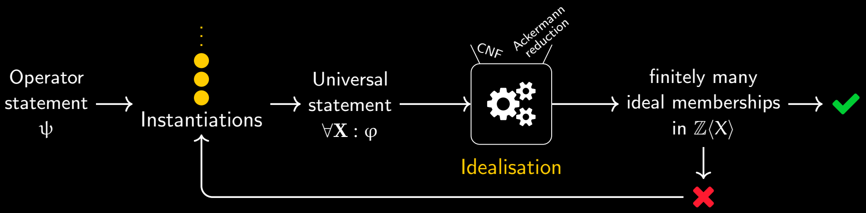

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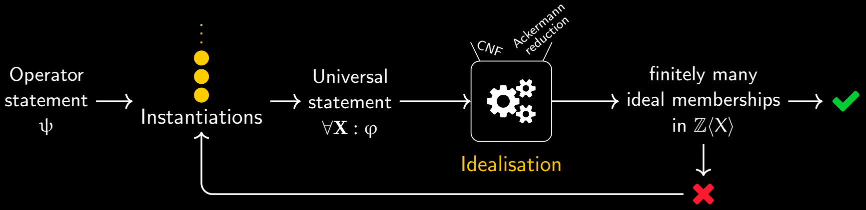

Theorem (H., Raab, Regensburger '22b)
An operator statement is universally true iff the procedure terminates and returns ✔

To treat all operator statements $\rightsquigarrow$ combine with Herbrand's theorem
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(d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.

20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then

(a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.

(b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.

(c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.

(d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.

(e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.

(f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.

22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^* A_j = 0$, and $A_i A_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:

(a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.

(b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.

(c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.

(d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.

(e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.

26. $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then

$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^*\\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^*\end{bmatrix},$$

where

$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Applications

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Applications

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  - they: *We use [. . .] decompositions of Hilbert spaces*
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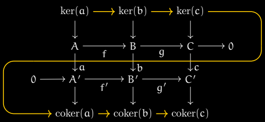
  - software used to find minimal assumptions
- Diagram lemmas (Five lemma, Nine lemma, Snake lemma,...)



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operator_gb

$$
\begin{aligned}
= \text{ nc Gröbner bases} + \text{certified ideal membership } &+ \text{ ideal arithmetic} \\
&+ \text{ heuristics} \\
&+ \text{ operator auxiliaries}
\end{aligned}
$$

operator_gb

= nc Gröbner bases + certified ideal membership + ideal arithmetic

Opal, Bergman, Magma, NCAlgebra (Mathematica), GAP, NCPoly (ApCoCoA)

Letterplace (Singular)

+ heuristics

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Foundation: efficient noncommutative F4 algorithm

Requires: fast monomial comparisons + fast (sparse) linear algebra

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monomials = strings $\rightsquigarrow$ efficient multi-pattern string matching

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signature_gb = Noncommutative signature Gröbner bases

Signature Gröbner bases

Observation Lots of redundant operations in GB computations

Goal Detect these operations!

$f = \sum a_{i,j} \cdot f_i \cdot b_{i,j}$

$\sigma = \mathrm{lt}(\sum a_{i,j} \cdot \varepsilon_i \cdot b_{i,j})$

Idea Add "birth certificate" to polynomials $f \rightsquigarrow (f, \sigma)$

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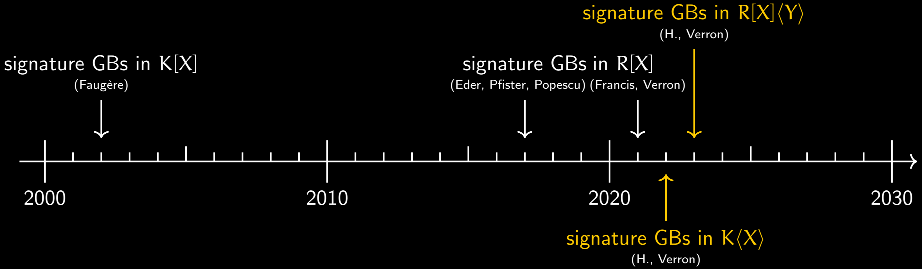

H., Verron. *Signature Gröbner bases, bases of syzygies and cofactor reconstruction in the free algebra*. In: Journal of Symbolic Computation. 2022.

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In the noncommutative setting...

...many things are similar (basic definitions, algorithm blueprint)

...many things are very different (trivial syzygies, handling of S-polynomials, decoupling selection strategy from signature order)

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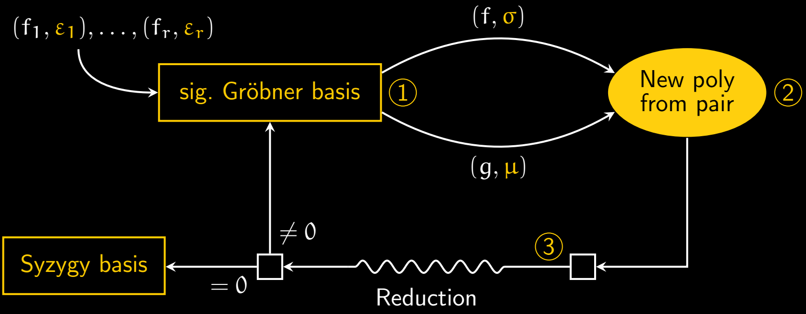
Sig-based Buchberger's algorithm



1. Selection: fair strategy *"Every S/G-poly is selected eventually"*
2. Construction: regular S/G-polynomials
3. Reduction: regular sig-reductions

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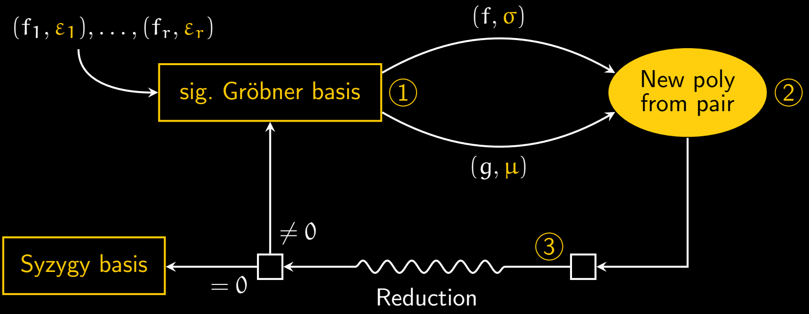

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Theorem (H., Verron '22, '23b)
This enumerates a (possibly infinite) sig. GB and syzygy basis

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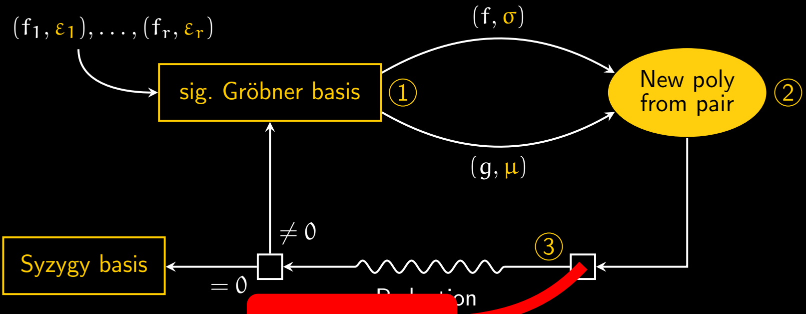

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operator_gb **vs.** signature_gb

Hilbert's 24th problem

"The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs.

- David Hilbert, ∼1900.

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operator statement ⇜⇝ proof

↕ ↕

ideal membership ⇜⇝ cofactor representation

$f \in (f_1, \ldots, f_r)$ $f = \sum_{i,j=1}^{N} a_{i,j} \cdot f_i \cdot b_{i,j}$

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Can we decide whether a cofactor representation exists?

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Can we decide whether a cofactor representation exists? – No.

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operator statement $\leftrightsquigarrow$ proof

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Can we decide whether a cofactor representation of length $\leqslant N$ exists? – Yes!

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Theorem (H., Verron '23a)

In a (minimal) cofactor representation

$$\max_{i,j} \deg(a_{i,j} \cdot f_i \cdot b_{i,j}) \leqslant \text{poly}(N, \deg(f), \deg(f_i)).$$

H., Verron. *Short proofs of ideal membership*. preprint. 2023.

Short proofs in practice

Theorem (Djordjević, Dinčić '09)

A, B matrices such that AB exists.

$$B^\dagger(ABB^\dagger)^\dagger \;=\; (A^\dagger AB)^\dagger A^\dagger \;=\; B^\dagger A^\dagger \quad\Rightarrow\quad (AB)^\dagger \;=\; B^\dagger A^\dagger$$

$$\rightsquigarrow \quad (ab)^\dagger - b^\dagger a^\dagger \in (f_1, \ldots, f_{44})$$

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Classical Gröbner bases: 1203 terms (~16 pages)

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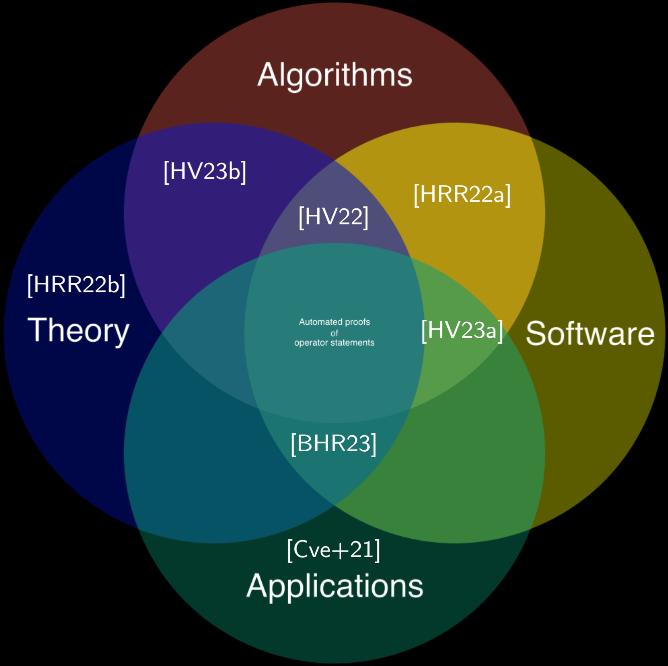
Classical Gröbner bases: 1203 terms (~16 pages)

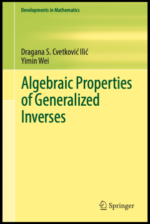
New approach:

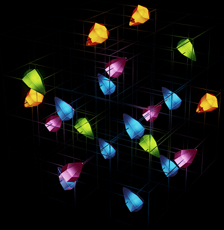
$$\begin{aligned}(ab)^\dagger - b^\dagger a^\dagger = \; & f_{21} - f_{10} + b^\dagger f_{14} - f_{12}(ab)^\dagger - b^\dagger(abb^\dagger)^\dagger f_{11} + b^\dagger(abb^\dagger)^\dagger f_{15} \\ & + (a^\dagger ab)^\dagger a^\dagger f_9(ab)^\dagger - b^* f_{23}((ab)^\dagger)^*(ab)^\dagger - f_{21}ab(ab)^\dagger + f_{22}ab(ab)^\dagger \\ & - f_{39}(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger + b^\dagger(abb^\dagger)^\dagger((abb^\dagger)^\dagger)^*(b^\dagger)^* f_{31} \\ & - b^\dagger f_{14} d^* b^*(a^\dagger)^* + (a^\dagger ab)^\dagger a^\dagger abf_{12}(ab)^\dagger \\ & - b^\dagger(abb^\dagger)^\dagger f_{15}((ab)^\dagger)^* b^*(a^\dagger)^* + f_{20}b^*(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger \\ & + (a^\dagger ab)^\dagger a^\dagger abb^* f_{23}((ab)^\dagger)^*(ab)^\dagger\end{aligned}$$


Algorithms
[HV23b]
[HRR22a]
[HV22]
[HRR22b]
Theory
Automated proofs
of
operator statements
[HV23a]
Software
[BHR23]
[Cve+21]
Applications

Outlook

- Producing proof certificates
- More advanced computational techniques
  - Boolean abstraction (DPLL(T), CDCL(T))
  - Congruence closure
  - Unification
  - ⋮
- Further applications
  - Generalised inverses
  - Abelian categories
  - Group theory





- Constructing counterexamples
- Investigating structure of Gröbner bases