A semi-decision procedure for proving operator statements

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AADIOS @ ACA 2023
Warsaw, Poland, July 18th, 2023

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DISCRETE MATHEMATICS AND ITS APPLICATIONS
Series Editor KENNETH H. ROSEN

HANDBOOK OF LINEAR ALGEBRA

SECOND EDITION

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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Leslie Hogben

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5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \dots, 1/\sigma_r, 0, \dots, 0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}^\dagger_{mn} = \mathbf{0}_{nm}$ and $J^\dagger_{mn} = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$
   Then
   - (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   - (b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   - (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   - (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   - (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   - (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   - (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   - (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   - (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   - (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   - (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   - (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   - (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   - (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   - (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   - (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   - (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   - (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   - (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim\limits_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim\limits_{\alpha\to 0} (\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum\limits_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum\limits_{j=1}^{\infty} (I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim\limits_{E\to 0}(A + E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank(A)}$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
   where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
   where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Theory

- Model linear operators by noncomm. polynomials
- Correctness of first-order operator statements
  $\iff$
  nc ideal membership
- Approach is complete
  $\longrightarrow$ Semi-decision procedure

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Software

- SageMath package `operator_gb`*
- Noncomm. Gröbner bases
- Certified nc ideal membership
- Noncomm. ideal arithmetic
- Dedicated methods for proving operator statements

* available at https://github.com/ClemensHofstadler/operator_gb

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Automated proofs of operator statements

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$$= \quad \sum_{i=1}^{d} c_i \cdot \quad x_{i,1} \ldots x_{i,k_i}$$

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finite words over X

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$$= \quad \sum_{i=1}^{d} c_i \cdot \underbrace{\boxed{x_{i,1} \dots x_{i,k_i}}}_{\text{finite words over } X}$$

Multiplication $=$ Concatenation of words

$$(x_1 \dots x_k) \cdot (x'_1 \dots x'_l) \quad = \quad x_1 \dots x_k x'_1 \dots x'_l$$

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Two-sided ideals For $f_1, \ldots, f_r \in \mathbb{Z}\langle X\rangle$

$$(f_1, \ldots, f_r) = \left\{ \sum_i \sum_j a_{i,j} \cdot f_i \cdot b_{i,j} \;\middle|\; a_{i,j}, b_{i,j} \in \mathbb{Z}\langle X\rangle \right\}$$

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Fact Ideal membership problem $f \overset{?}{\in} (f_1, \ldots, f_r)$ is semi-decidable (e.g., using Gröbner bases)

Operator statements

Operators

- $0, a, b, c, \ldots$
- $s + t,\ s \cdot t,\ f(t_1, \ldots, t_n)$

Linearity = abelian (partial) addition + assoc. (partial) mult. + dist.

Operator statements

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$^*,\ \cdot^{\mathsf{T}},\ \|\cdot\|,\ \otimes, \dots$

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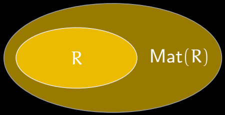

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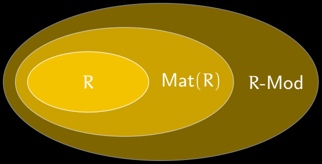

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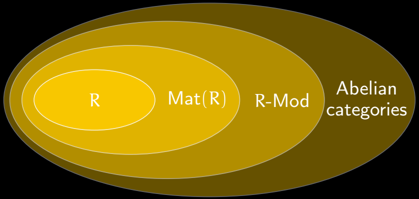

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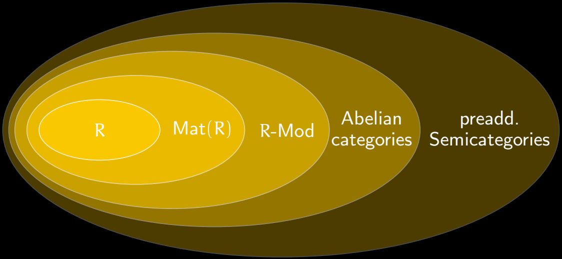

R

Mat(R)

R-Mod

Abelian categories

preadd. Semicategories

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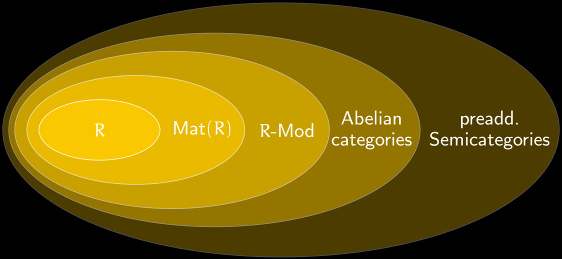
$$s = t,\quad \neg\,\varphi,\quad (\varphi \wedge \psi),\quad (\varphi \vee \psi),\quad (\varphi \Rightarrow \psi),\quad \exists\, x : \varphi,\quad \forall\, x : \varphi$$



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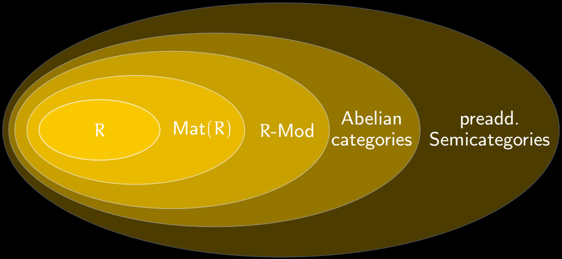
Definition An operator statement is universally true if it follows from linearity



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Definition An operator statement is universally true if it follows from linearity

- Fact: Determining universal truth is not decidable
  $\Rightarrow$ Algorithm that terminates on all inputs cannot exist
- Best we can hope for: (effective) semi-decision procedure
  $\longrightarrow$ Can be obtained using computer algebra

Determining universal truth

Idea Translate universal truth of formula into polynomial predicate

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Quasi-identities

- Classical case of quasi-identities well studied (Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$$\forall \mathbf{X} : \bigwedge_{j=1}^{m} A_j = B_j \Rightarrow P = Q$$

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Strategy

- Interpret each operator as polynomial in $\mathbb{Z}\langle \mathbf{X} \rangle$ and reformulate each identity $L = R$ as polynomial $L - R$

$$\text{e.g.,} \quad AB = BA \quad \rightsquigarrow \quad ab - ba \in \mathbb{Z}\langle a, b \rangle$$

- "Being a consequence" ($\Rightarrow$) translates into ideal membership

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Theorem

$$\forall \mathbf{X} : \bigwedge_{j=1}^{m} A_j = B_j \Rightarrow P = Q \qquad \text{iff} \qquad p - q \in (a_1 - b_1, \dots, a_m - b_m)$$

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Recall: B is Moore-Penrose inverse of A if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

Claim If B and C satisfy these identities, then $B = C$

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sage: from operator_gb import *
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        + b*(-a*c + c_adj*a_adj) - b*(-a*c + c_adj*a_adj)*b_adj*a_adj
        - (-b + b*a*b) + (-c*a + a_adj*c_adj)*b*a*c
        - (-a_adj + a_adj*c_adj*a_adj)*b_adj*c + c*(-a + a*b*a)*c
        - (-b*a + a_adj*b_adj)*c + a_adj*c_adj*(-b*a + a_adj*b_adj)*c
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```

- Software produces cofactor representation
  (= certificate for ideal membership)
- Cofactor representation is algebraic proof requiring only linearity
  ⇒ Statement is proven in all settings where linearity holds

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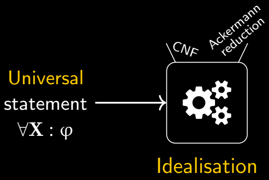
Universal statements



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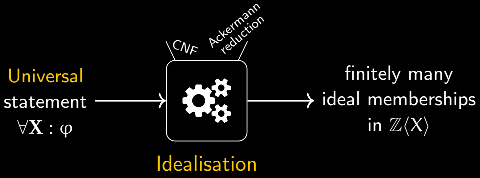

Universal statement $\forall \mathbf{X} : \varphi$ → Idealisation → finitely many ideal memberships in $\mathbb{Z}\langle X \rangle$

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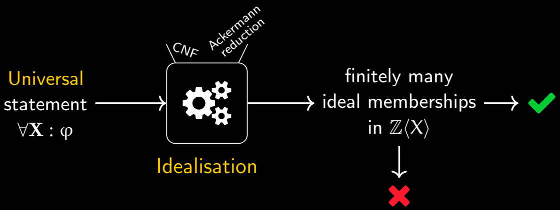


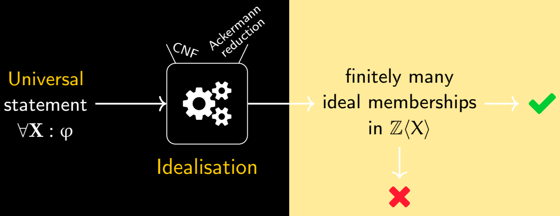
Theorem (H., Raab, Regensburger '22)
A universal statement is universally true iff its idealisation is true

Determining universal truth

Idea Translate universal truth of formula into polynomial predicate

Universal statements

effectively semi-decidable
using Gröbner bases



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5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where

$$\Sigma^\dagger = \mathrm{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$

5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem

$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$

6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote

$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

Then

   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.

   (b) $(\mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \mathrm{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.

10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,

   (a) if $A \in \mathbb{C}^{m\times n}$ ($m \ge n$) has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;

   (b) if $A \in \mathbb{C}^{m\times n}$ ($m \le n$) has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.

19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.

   (b) $\mathrm{rank}(A) = \mathrm{rank}(A^\dagger) = \mathrm{rank}(AA^\dagger) = \mathrm{rank}(A^\dagger A)$.

   (c) $\mathrm{rank}(I_n - A^\dagger A) = n - \mathrm{rank}(A)$.

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20. $AA^\dagger = \mathrm{Proj}_{\mathrm{range}(A)}$; $A^\dagger A = \mathrm{Proj}_{\mathrm{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then

   (a) $\mathrm{range}(A) = \mathrm{range}(AA^*) = \mathrm{range}(AA^\dagger)$.

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   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.

   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.

   (e) $\mathrm{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.

   (f) $\mathrm{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.

22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:

   (a) $\mathrm{range}(BB^*A^*) \subseteq \mathrm{range}(A^*)$ and $\mathrm{range}(A^*AB) \subseteq \mathrm{range}(B)$.

   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.

   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.

   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.

   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.

26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim\limits_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim\limits_{\alpha\to 0} (\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum\limits_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum\limits_{j=1}^{\infty} (I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim\limits_{E\to 0}(A + E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\mathrm{rank}(A + E) = \mathrm{rank(A)}$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m, n\}$. Suppose that A can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{r\times r}$ and $\mathrm{rank}(A_{11}) = r$. Then

$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$

where

$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

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Strategy

1. Derive explicit expression for X
2. Reformulate statement as a universal statement
3. Prove by verifying ideal membership

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sage: assumptions =  [a - p*a_adj*a,...]
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$\Rightarrow$ $X = A^*QP^*$ is MP-inverse of A

(can be certified using the software)

Existential statements

In the previous example, we found a suitable expression.

Question Was this just luck?

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An existential statement is universally true if and only if explicit expressions exist and can be constructed as polynomial expressions in terms of the basic operators appearing in the statement.

Existential statements

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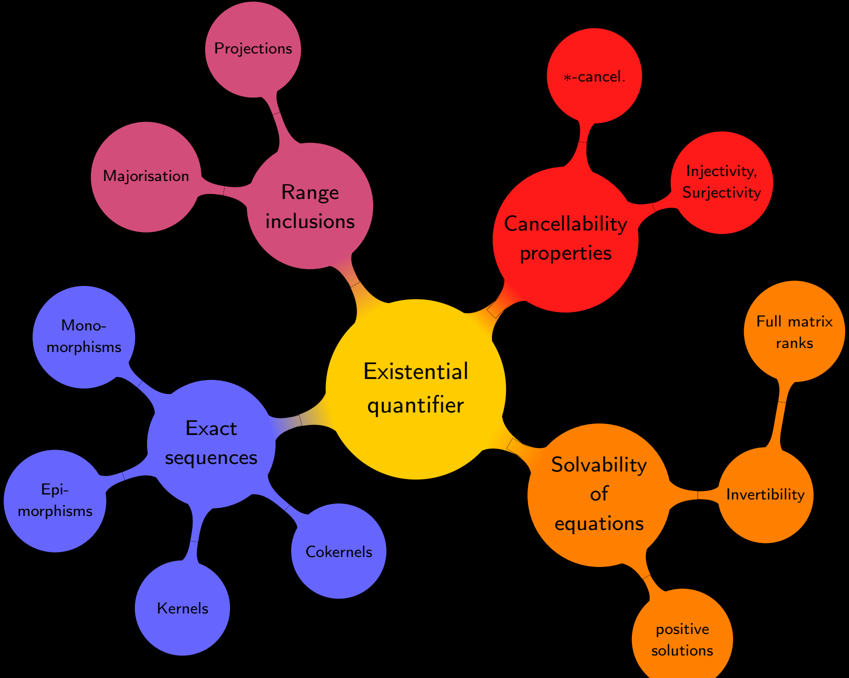
Reason Herbrand's theorem (Herbrand '30)

An existential statement is universally true if and only if explicit expressions exist and can be constructed as polynomial expressions in terms of the basic operators appearing in the statement.

- Enumerating all possible expressions is hopeless
- Requires good heuristics → provided by computer algebra
- Several heuristics implemented in `operator_gb`
  (ansatz, variable elimination, ideal/subalgebra intersections,. . .)

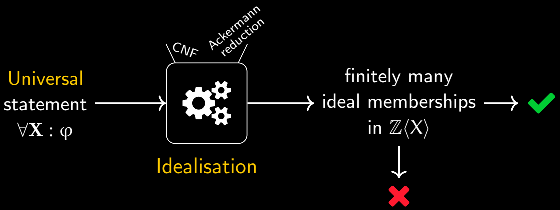
Existential
quantifier


Existential quantifier
Range inclusions
Projections
Majorisation
Cancellability properties
∗-cancel.
Injectivity, Surjectivity
Exact sequences
Mono-morphisms
Epi-morphisms
Kernels
Cokernels
Solvability of equations
Invertibility
Full matrix ranks
positive solutions

Determining universal truth

Idea Translate universal truth of formula into polynomial predicate

Universal statements



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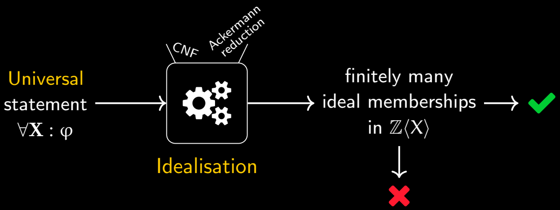

Universal statement $\forall \mathbf{X} : \varphi$ → Idealisation → finitely many ideal memberships in $\mathbb{Z}\langle X\rangle$ → ✓

↓

✗

Theorem (H., Raab, Regensburger '22)
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To treat all operator statements ⇝ combine with Herbrand's theorem
\+ Heuristics

Determining universal truth

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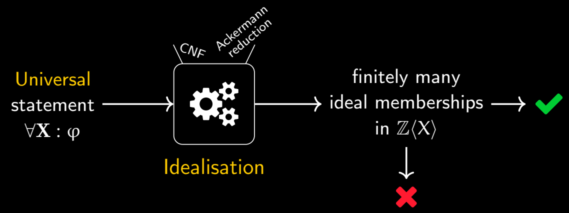
General operator statements



CNF

Ackermann reduction

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↓

✗

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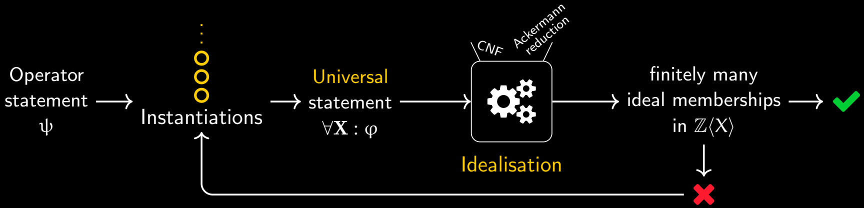

Operator statement ψ → Instantiations → Universal statement $\forall \mathbf{X} : \varphi$ → Idealisation → finitely many ideal memberships in $\mathbb{Z}\langle X \rangle$ → ✓

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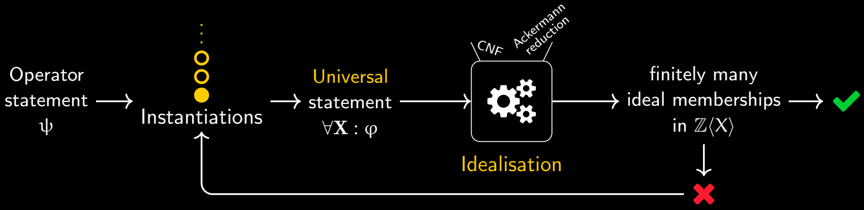

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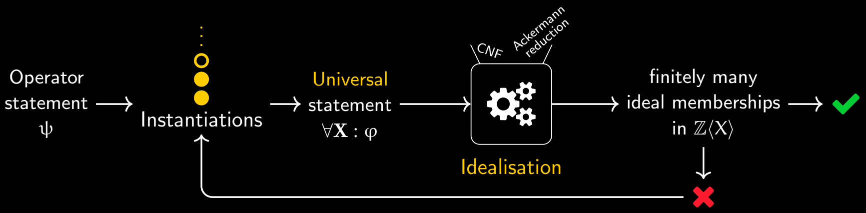

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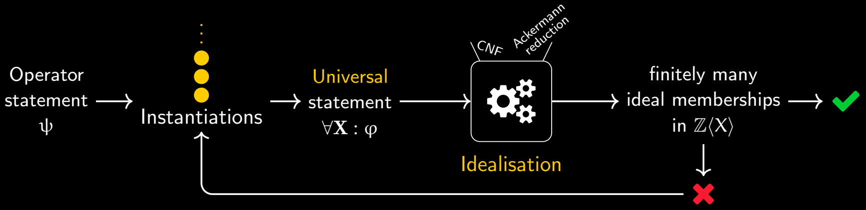

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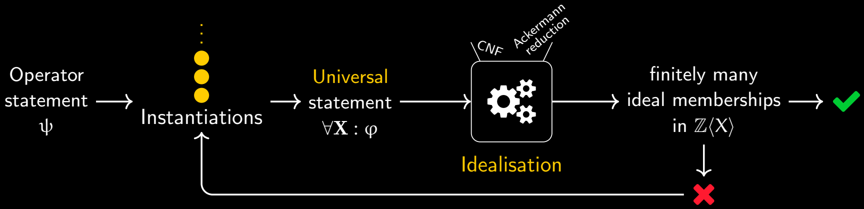

Operator statement ψ → Instantiations → Universal statement $\forall \mathbf{X} : \varphi$ → Idealisation (CNF, Ackermann reduction) → finitely many ideal memberships in $\mathbb{Z}\langle X\rangle$ → ✓

✗ → Instantiations

Theorem (H., Raab, Regensburger '22)
An operator statement is universally true iff the procedure terminates and returns ✓

To treat all operator statements $\rightsquigarrow$ combine with Herbrand's theorem
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   - (e) ✓ $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   - (f) ✓ $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. ✓ If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. ✓ If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. ✓ $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. ✓ [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   - (a) ✓ $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   - (b) ✓ $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   - (c) ✓ $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   - (d) ✓ $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   - (e) ✓ $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. ✓ $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. ✗ $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. ✗ $A^\dagger = \displaystyle\sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. ✗ (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. ✓ Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Applications

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- Diagram lemmas (Five lemma, Nine lemma, Snake lemma,. . .)

$$\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll] & \\
 & A \arrow[r, "f"'] \arrow[d, "a"] & B \arrow[r, "g"'] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) &
\end{tikzcd}$$

`operator_gb`

= nc Gröbner bases + certified ideal membership + ideal arithmetic

\+ heuristics

\+ operator auxiliaries

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= **nc Gröbner bases** + certified ideal membership + ideal arithmetic

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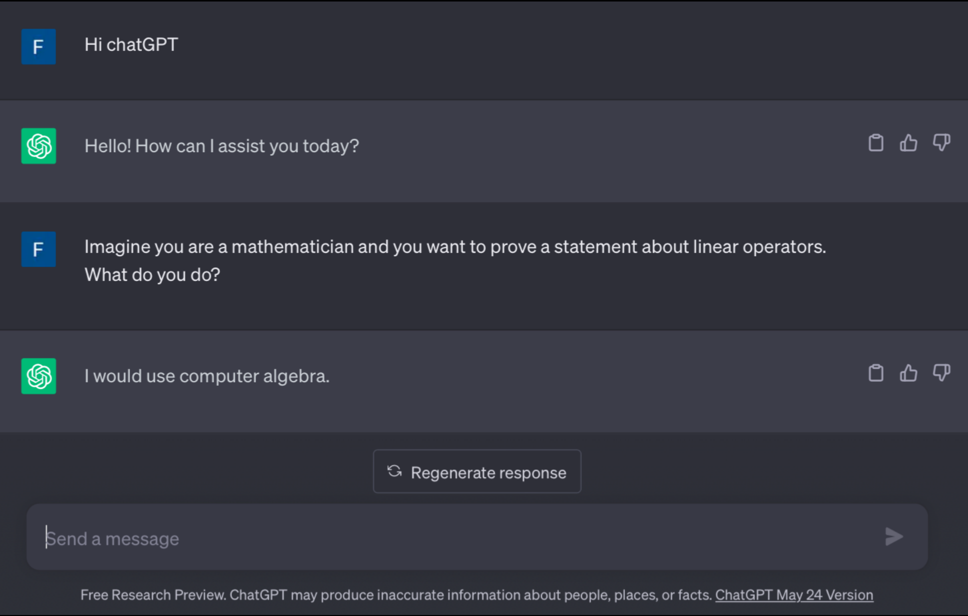
Foundation: efficient noncommutative F4 algorithm

Requires: fast monomial comparisons + fast (sparse) linear algebra

Realised via:

- Monomials are represented by (encoded) strings ⇝ exploit efficient multi-pattern string matching algorithms (Aho-Corasick algo. '75) in C (pyahocorasick)
- Dedicated (sparse) linear algebra routines in C (via Cython) exploiting structure of the matrices (Faugère-Lachartre elim. '10)
- Noncommutative signature-based techniques (in the making)



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Hello! How can I assist you today?

Imagine you are a mathematician and you want to prove a statement about linear operators. What do you do?

I would use computer algebra.

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