Signature Gröbner Bases in Free Algebras over Rings

Clemens Hofstadler[1], Thibaut Verron[2]

ISSAC 2023
Tromsø, Norway, July 25th, 2023

1. Institute of Mathematics, University of Kassel, Germany
2. Institute for Algebra, Johannes Kepler University, Linz, Austria

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FWF Der Wissenschaftsfonds.

The (vast) world of Gröbner theories

$\mathbb{K}[X]$
[Buchberger '65]

The (vast) world of Gröbner theories

ambiguities

$K\langle X\rangle$

[Bergman '78, Mora '85, ...]

$K[X]$

[Buchberger '65]

$R[X]$

G-polynomials

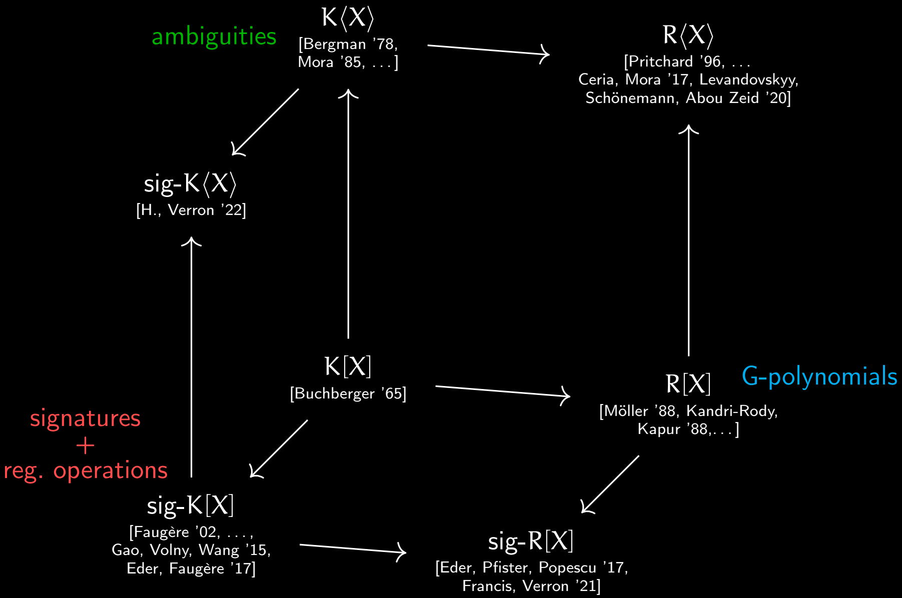
[Möller '88, Kandri-Rody, Kapur '88,...]

signatures
+
reg. operations

sig-$K[X]$

[Faugère '02, ..., Gao, Volny, Wang '15, Eder, Faugère '17]

The (vast) world of Gröbner theories



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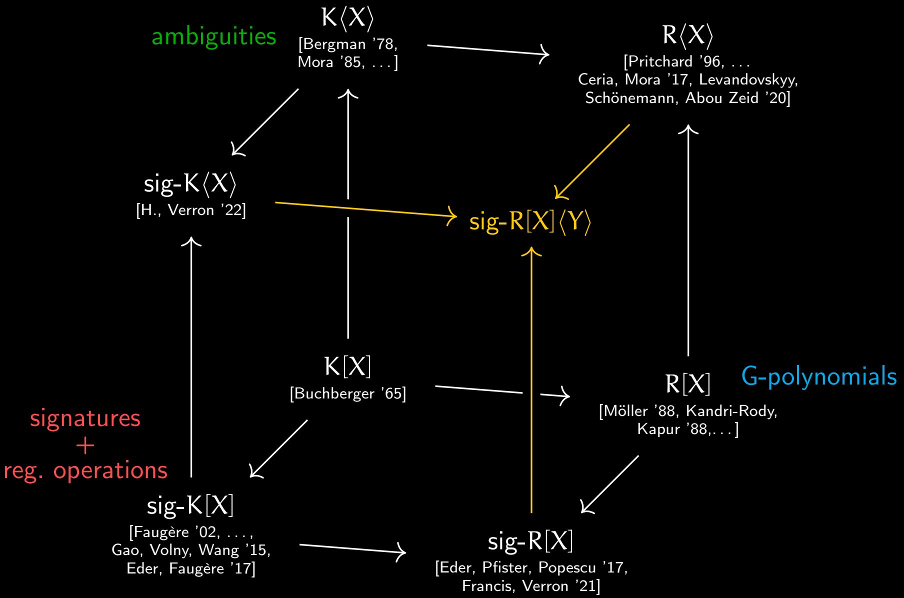

Signature Gröbner Bases in ~~Free~~ Mixed Algebras over Rings

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The mixed algebra

[Mikhalev, Zolotykh '98]

Mixed algebra $\quad = \quad R[X]\langle Y\rangle$

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comm. PID

The mixed algebra

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Mixed algebra $=$ $R[X]\langle Y\rangle$

$$\{x^{a} = x_1^{a_1} \ldots x_k^{a_k} \mid a \in \mathbb{N}^k\}$$

The mixed algebra

[Mikhalev, Zolotykh '98]

Mixed algebra $=$ $R[X]\langle Y\rangle$

$\{y_{i_1} \dots y_{i_l} \mid l \in \mathbb{N}\}$

The mixed algebra

[Mikhalev, Zolotykh '98]

$$\text{Mixed algebra} = R[X]\langle Y\rangle \simeq R\langle X, Y\rangle \big/ \left(xz - zx \mid x \in X,\, z \in X \cup Y\right)$$

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(Mixed) polynomial $= \sum_{i=1}^{d} c_i \cdot \mathbf{x}^{\mathbf{a}_i} w_i$

Multiplication : $\mathbf{x}^{\mathbf{a}} v \cdot \mathbf{x}^{\mathbf{b}} w = \mathbf{x}^{\mathbf{a}+\mathbf{b}} vw$

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Two-sided ideals For $f_1, \ldots, f_r \in R[X]\langle Y \rangle$

$$(f_1, \ldots, f_r) = \left\{ \sum_i \sum_j p_{i,j} \cdot f_i \cdot q_{i,j} \;\middle|\; p_{i,j},\ q_{i,j} \in R[X]\langle Y \rangle \right\}$$

Why the mixed algebra?

In the free algebra. . .

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Solution New setting for signatures $\rightarrow$ Mixed algebra

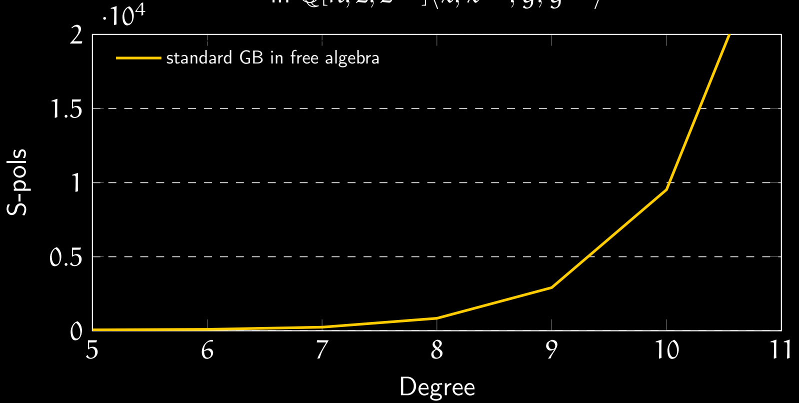
Experiments

Prototype implementation for SageMath (when R is a field).

Consider homogenisation of the discrete Heisenberg group

$$\langle x, y, z \mid z = xyx^{-1}y^{-1}, xz = zx, yz = zy \rangle .$$

in $\mathbb{Q}[h, z, z^{-1}]\langle x, x^{-1}, y, y^{-1} \rangle$



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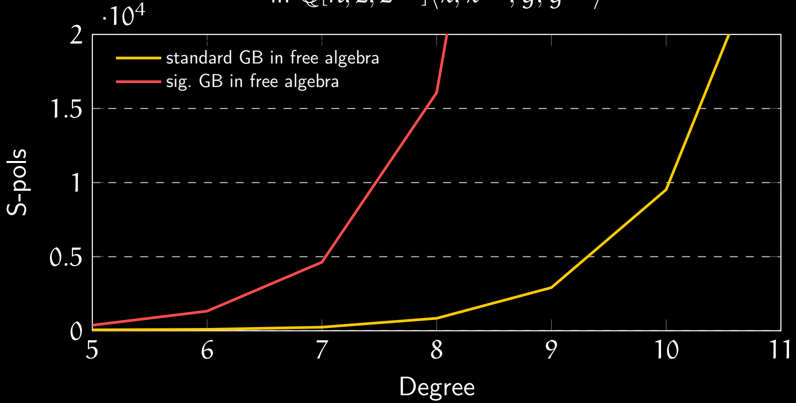

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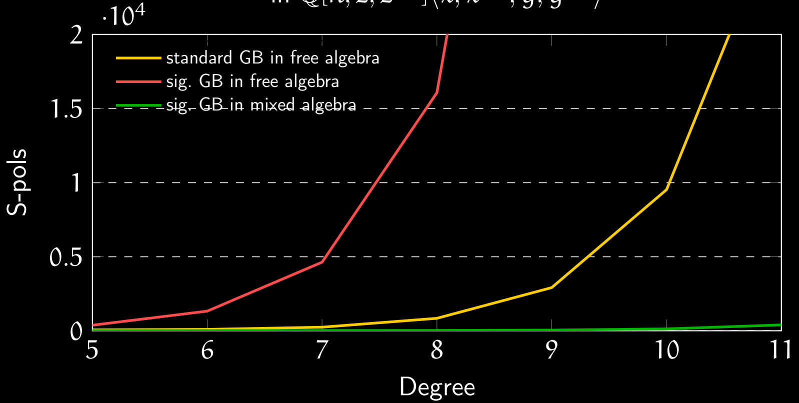

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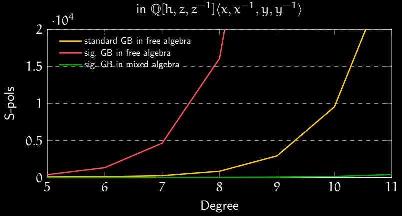

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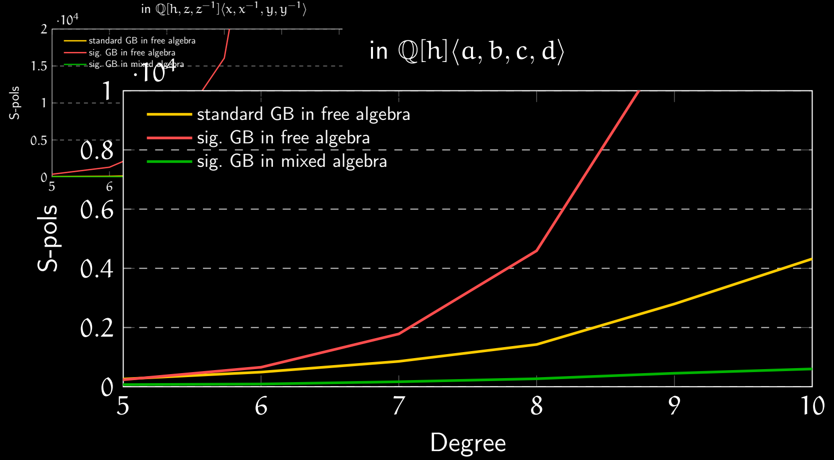

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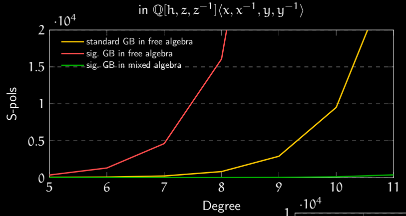

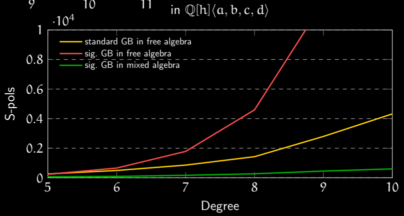

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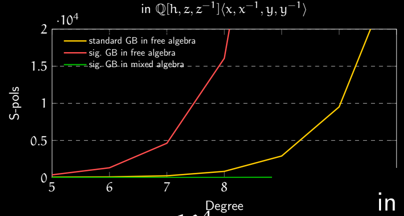

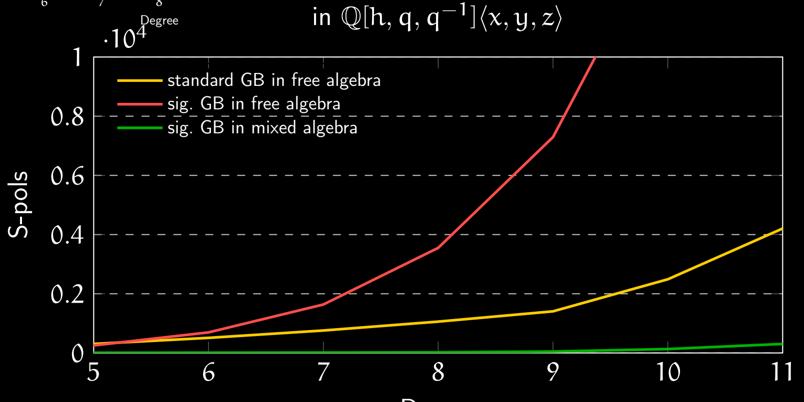

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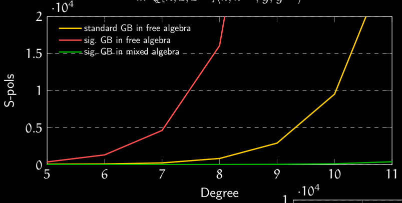
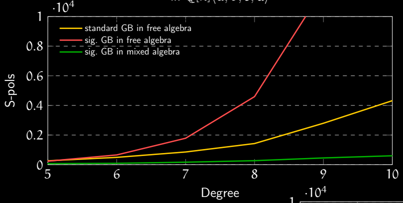


in $\mathbb{Q}[h]\langle a, b, c, d\rangle$



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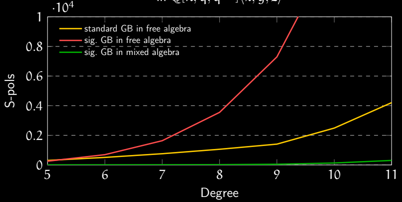

The mixed algebra

What's standard?

- Monomial order and leading coefficient/monomial/term

$$\Rightarrow \quad f = \underbrace{\overbrace{c \cdot x^a w}^{\mathrm{lt}(f)}}_{\mathrm{lc}(f)\ \ \mathrm{lm}(f)} + \text{ smaller terms}$$

- Reductions: if $\mathrm{lt}(f) = s \cdot \mathrm{lt}(g) \cdot t$, then $f \to f - sgt$

- Gröbner bases: $G \subseteq I \trianglelefteq R[X]\langle Y \rangle$ is

$$\text{Gröbner basis} \quad \text{if} \quad f \xrightarrow{*}_G 0 \text{ for all } f \in I$$

The mixed algebra

What's not so standard?

…because of $\langle Y \rangle$

…because of $[X]$

…because of R

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- Most ideals do not have finite Gröbner basis

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- lcm of leading monomials not unique

  For yzy and yz both $yzyz$ and yzy are minimal multiples.

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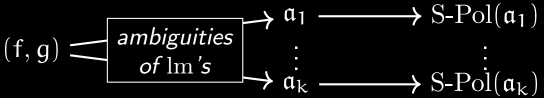

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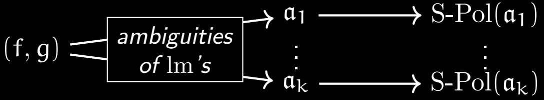

- Non-minimal ambiguities corresponding to

$$\mathrm{lm}(f) \;\blacksquare\; \mathrm{lm}(g) \qquad \forall \blacksquare \in \langle Y \rangle$$

  $\Rightarrow$ two elements have infinitely many S-polynomials ... because of $[X]$

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  For yzy and yz both $yzyz$ and yzy are minimal multiples. (ambiguity $\mathfrak{a}$)

  $$(f, g) \to \text{ambiguities of } \mathrm{lm}\text{'s} \to \begin{cases} \mathfrak{a}_1 \longrightarrow \text{S-Pol}(\mathfrak{a}_1) \\ \vdots \qquad\qquad \vdots \\ \mathfrak{a}_k \longrightarrow \text{S-Pol}(\mathfrak{a}_k) \end{cases}$$

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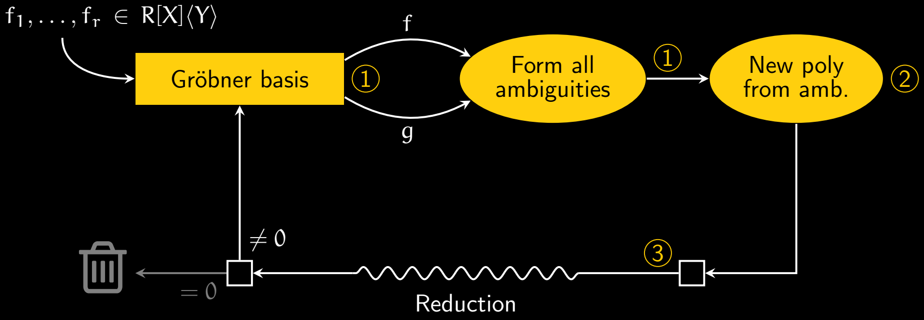
- G-polynomials to introduce new leading coefficients
  - $\text{S-Pol}(\mathfrak{a})$: $\mathrm{lm}(\text{S-Pol}) =$ new and $\mathrm{lc}(\text{S-Pol}) = ?$
  - $\text{G-Pol}(\mathfrak{a})$: $\mathrm{lm}(\text{G-Pol}) =$ old and $\mathrm{lc}(\text{G-Pol}) =$ minimal

  … because of R

Buchberger's algorithm



1. Selection: fair strategy *"Every S/G-poly is selected eventually"*
2. Construction: $\text{S-Pol}(\mathfrak{a})$ and $\text{G-Pol}(\mathfrak{a})$ from ambiguity $\mathfrak{a}$
3. Reduction: if $\text{lt}(f) = s \cdot \text{lt}(g) \cdot t$, then $f \to f - sgt$

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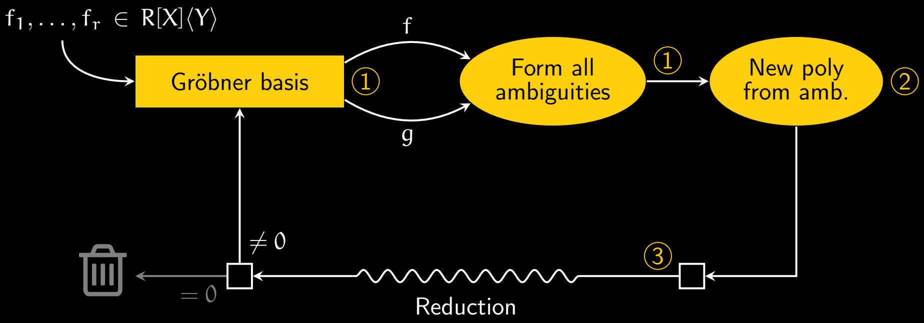

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Theorem [Mikhalev, Zolotykh '98] This enumerates a (possibly infinite) GB

Signatures

Idea Add module perspective to polynomial computations

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$$\mathcal{A} = R[X]\langle Y\rangle$$

mixed algebra

$$I = (f_1, \ldots, f_r)$$

$$f = \sum_i c_i \cdot x^{a_i} v_i f_{j_i} w_i \qquad v_i, w_i \in \langle Y\rangle$$

$\mathrm{lt}(f)$ = largest term in f

leading term

Signatures

Idea Add module perspective to polynomial computations

$$\Sigma = \bigoplus_{i=1}^{r} \mathcal{A} \otimes_{R[X]} \mathcal{A}$$

free $\mathcal{A}$-bimodule

$$\mathcal{A} = R[X]\langle Y \rangle$$

mixed algebra

$$\varepsilon_1, \ldots, \varepsilon_r$$

$$I = (f_1, \ldots, f_r)$$

$$\alpha = \sum_i c_i \cdot \mathbf{x}^{a_i} v_i \varepsilon_{j_i} w_i$$

$$f = \sum_i c_i \cdot \mathbf{x}^{a_i} v_i f_{j_i} w_i \qquad v_i, w_i \in \langle Y \rangle$$

$\mathrm{sig}(\alpha)$ = largest term in α

signature

$\mathrm{lt}(f)$ = largest term in f

leading term

Signatures

Idea Add module perspective to polynomial computations

$\Sigma = \bigoplus_{i=1}^{r} \mathcal{A} \otimes_{R[X]} \mathcal{A}$	$\mathcal{A} = R[X]\langle Y \rangle$
free $\mathcal{A}$-bimodule	mixed algebra
$\varepsilon_1, \ldots, \varepsilon_r$	$I = (f_1, \ldots, f_r)$
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$\mathrm{sig}(\alpha) =$ largest term in α	$\mathrm{lt}(f) =$ largest term in f
signature	leading term

Note: $x\alpha = \alpha x$ but $y\alpha \neq \alpha y \Rightarrow$ signature sees commutators

Signatures

Idea Add module perspective to polynomial computations

$\Sigma = \bigoplus_{i=1}^{r} \mathcal{A} \otimes_{R[X]} \mathcal{A}$ free $\mathcal{A}$-bimodule	$\mathcal{A} = R[X]\langle Y \rangle$ mixed algebra
$\varepsilon_1, \ldots, \varepsilon_r$	$I = (f_1, \ldots, f_r)$
$\alpha = \sum_i c_i \cdot x^{a_i} v_i \varepsilon_{j_i} w_i \quad \longmapsto$	$f = \sum_i c_i \cdot x^{a_i} v_i f_{j_i} w_i \qquad v_i, w_i \in \langle Y \rangle$
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$\mathcal{A}$-bimodule hom.: $\quad \bar{\cdot} : \Sigma \to I, \quad \varepsilon_i \mapsto f_i$

Signatures

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Sig-based algorithms work with pairs $(f, \mathrm{sig}(\alpha))$ where $\overline{\alpha} = f$

Regular operations

$\sigma \succ \mu \quad \Rightarrow \quad (f, \sigma) \pm (g, \mu) \; = \; (f \pm g, \sigma)$

$\Rightarrow$ regular reductions, regular S/G-polynomials

$(f, \sigma) \to (f - sgt, \sigma)$

Signature based algorithms

To add signatures to the classical Buchberger algorithm, we have to...

1. Replace polynomials by signature polynomials
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Signature based algorithms compute...

- Signature Gröbner basis
  - Definition just like in the commutative case
  - Signature GBs are in particular GBs

    G signature GB $\Rightarrow$ $\{ f \mid (f, \sigma) \in G \}$ GB
- Gröbner basis of the syzygy module

Elimination criteria

Characterisation of sig. GB G and syzygy GB H via cover criterion

[Gao, Volny, Wang '15, Francis, Verron, '21, H., Verron '23]

Regular ambiguity $\mathfrak{a}$ covered by (G, H) if

"$\text{S-Pol}(\mathfrak{a})$ *not minimal among elements with* $\text{sig}(\text{S-Pol})$ *in span of* $G \cup H$"

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Cover criterion

all regular ambiguities of G covered $\quad + \quad$ enough $\mathrm{G\text{-}Pol}$

$\Rightarrow \quad G$ sig. GB, $\quad H$ syz. GB

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Facts

- Ambiguity $\mathfrak{a}$ is covered by $\text{S-Pol}(\mathfrak{a})$...
- ...but can also be covered by other elements $\Rightarrow$ elimination criterion for S-Pol (includes syzygy, singular, and F5 criterion)

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- G-Pol redundant if reducible

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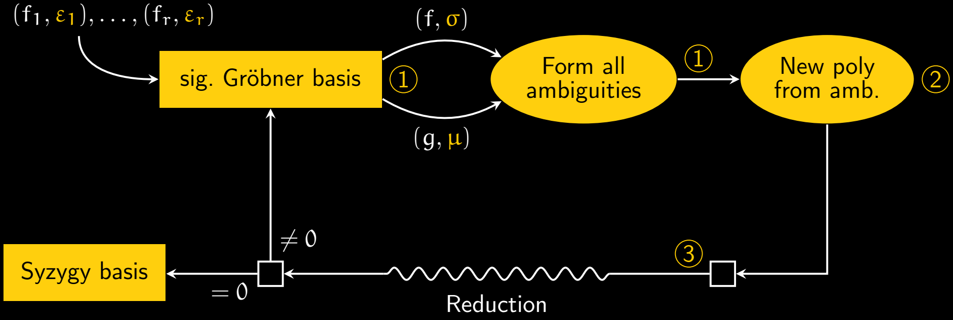
Facts

- Ambiguity $\mathfrak{a}$ is covered by $\text{S-Pol}(\mathfrak{a})$...
- ...but can also be covered by other elements $\Rightarrow$ elimination criterion for S-Pol (includes syzygy, singular, and F5 criterion)
- G-Pol redundant if reducible
- Cover criterion decouples selection strategy from signature order
    - In the commutative case this is nice
    - In the noncommutative case this is essential

Sig-based Buchberger's algorithm



1. Selection: fair strategy *"Every S/G-poly is selected eventually"*
2. Construction: regular $\text{S-Pol}(\mathfrak{a})$ and $\text{G-Pol}(\mathfrak{a})$ from ambiguity $\mathfrak{a}$
3. Reduction: (regular) if $\mathrm{lt}(f) = s \cdot \mathrm{lt}(g) \cdot t$ and $\sigma \succ s \cdot \mu \cdot t$, then $(f, \sigma) \to (f - sgt, \sigma)$

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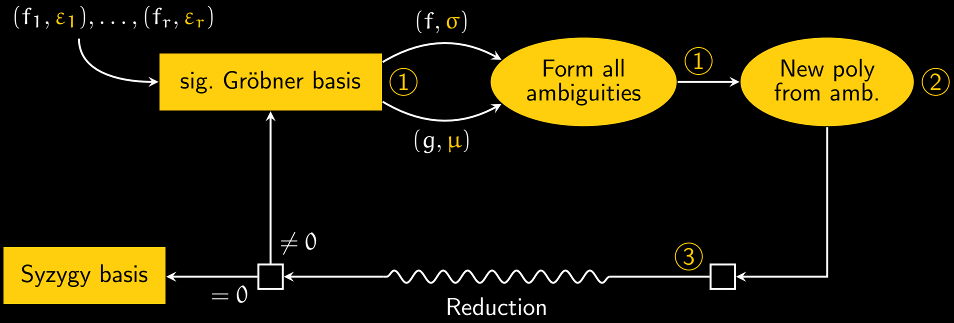

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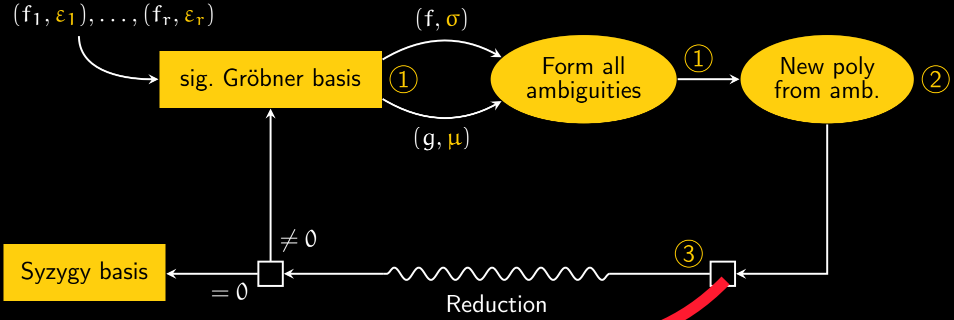


S-Pol($\mathfrak{a}$) and covered?
G-Pol($\mathfrak{a}$) and reducible?

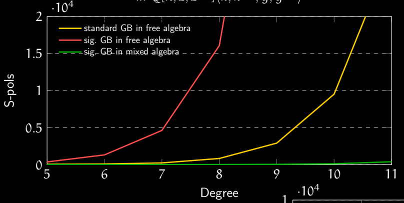
1. **Selection:** fai … *ly is selected eventually”*
2. **Construction:** regular S-Pol($\mathfrak{a}$) and G-Pol($\mathfrak{a}$) from ambiguity $\mathfrak{a}$
3. **Reduction:** (regular) if $\operatorname{lt}(f) = s \cdot \operatorname{lt}(g) \cdot t$ and $\sigma \succ s \cdot \mu \cdot t$, then $(f, \sigma) \to (f - sgt, \sigma)$

Theorem [H., Verron '23] This enumerates a (possibly infinite) sig. GB and syzygy basis

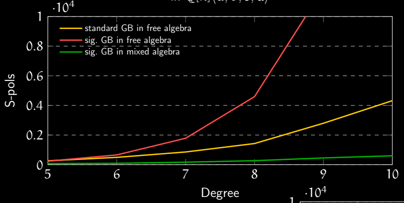
Experiments

Prototype implementation for SageMath (when R is a field).

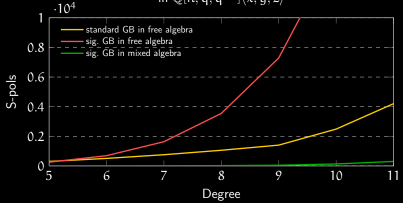
in $\mathbb{Q}[h, z, z^{-1}]\langle x, x^{-1}, y, y^{-1}\rangle$



in $\mathbb{Q}[h]\langle a, b, c, d\rangle$



in $\mathbb{Q}[h, q, q^{-1}]\langle x, y, z\rangle$



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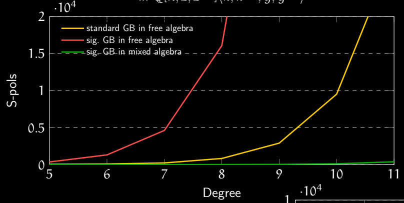

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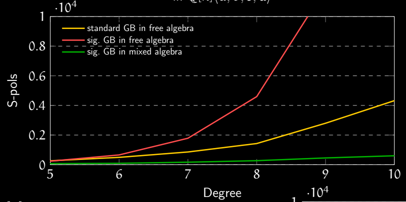
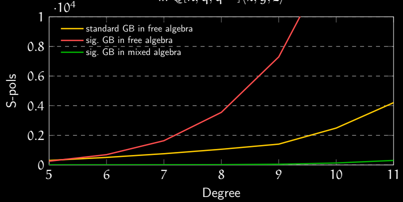


+ data that allows to. . .

- . . . obtain representations of ideal elements (almost) for free
- . . . compute minimal representations of ideal elements [H., Verron '23]

in $\mathbb{Q}[h, q, q^{-1}]\langle x, y, z\rangle$



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Hello! How can I assist you today?

F

Imagine you want to compute Gröbner bases in the free algebra but some of your variables are commutative. What do you do?

I would use signature Gröbner bases in the mixed algebra.

Regenerate response

Send a message

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