Automated proofs of operator statements

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DISCRETE MATHEMATICS AND ITS APPLICATIONS
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HANDBOOK OF LINEAR ALGEBRA

SECOND EDITION

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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Leslie Hogben

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5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1,\ldots,1/\sigma_r,0,\ldots,0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \ne \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \ne \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases}\alpha^{-1}, & \text{if } \alpha \ne 0,\\ 0, & \text{if } \alpha = 0.\end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1,\beta_2,\cdots,\beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger,\beta_2^\dagger,\cdots,\beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i,j = 1,\cdots,k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^*\\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^*\end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

In this paper we present new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. Some finite-dimensional results are extended to infinite-dimensional settings.

1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a) 1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.
Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.
Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.
Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.
Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.
Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.
In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.
Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

Theory

- Consider linear operators as symbolic expressions
- Correctness of first-order operator statements

  $\Longleftrightarrow$

  existence of cofactor representations
- Approach is complete

  $\longrightarrow$ Every true statement can be proven

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Software

- SageMath package `operator_gb`*
- Efficient open-source implementation
- Cofactor representations
- Dedicated methods for proving operator statements

*available at https://github.com/ClemensHofstadler/operator_gb

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Automated proofs of operator statements

Operator statements

Operators

- $0, a, b, c, \ldots$
- $s + t,\ s \cdot t,\ f(t_1, \ldots, t_n)$

Linearity = abelian (partial) addition + assoc. (partial) mult. + dist.

Operator statements

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$^*,\ \cdot^{\mathsf{T}},\ \|\cdot\|,\ \otimes, \ldots$

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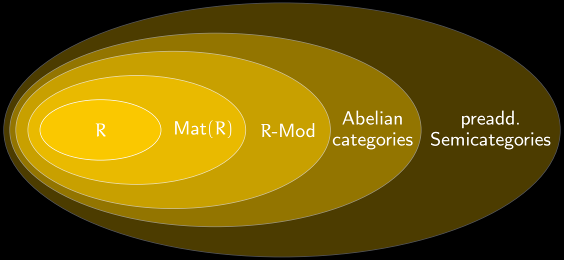

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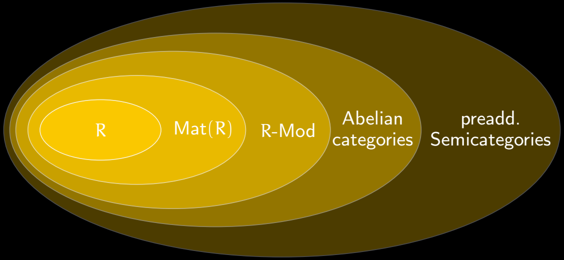
Operator statements

$$s = t, \quad \neg\,\varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad (\varphi \Rightarrow \psi), \quad \exists\, x : \varphi, \quad \forall\, x : \varphi$$



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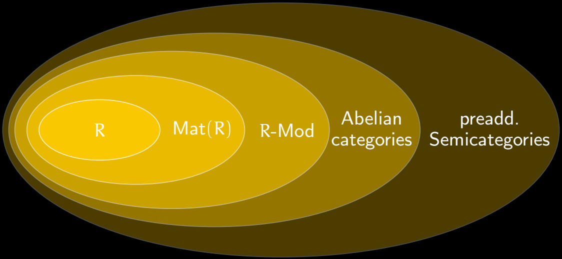
Definition An operator statement is universally true if it follows from linearity



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- Fact: Determining universal truth is not decidable
  $\Rightarrow$ Algorithm that terminates on all inputs cannot exist

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Definition An operator statement is universally true if it follows from linearity

- Fact: Determining universal truth is not decidable
  $\Rightarrow$ Algorithm that terminates on all inputs cannot exist
- Best we can hope for: (efficient) semi-decision procedure

Operator statements

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Definition An operator statement is universally true if it follows from linearity

Theorem (H., Raab, Regensburger '22)
There exists a semi-decision procedure for determining universal truth of operator statements based on symbolic computations.
It can be realised efficiently using computer algebra.

Toy example: *"The Moore-Penrose inverse is unique"*

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Recall: B is Moore-Penrose inverse of A if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

Claim If B and C satisfy these identities, then $B = C$

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Proof

$$B = BAB = BACAB = \ldots = C$$

A different point of view

$$L = R \iff L - R = 0$$

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Claim If B and C satisfy these identities, then $B = C$

Proof $$B = BAB = BACAB = \ldots = C$$

A different point of view

$$L = R \quad \Longleftrightarrow \quad L - R$$

$$L = M = R \quad \Longleftrightarrow \quad L - R = (L - M) + (M - R)$$

Toy example: *"The Moore-Penrose inverse is unique"*

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$$L = R \iff L - R$$
$$L = M = R \iff L - R = (L - M) + (M - R)$$

Theorem (Raab, Regensburger, Hossein Poor '21)

$$\bigwedge_{i=1}^{m} A_i = B_i \Rightarrow L = R \qquad \text{iff} \qquad L - R = \sum_j c_j \cdot P_j \left(A_{i_j} - B_{i_j}\right) Q_j$$

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$$L = R \iff L - R$$
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- "cofactor representation"
- computable with computer algebra

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Proof Using our software package `operator_gb`...

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Proof Using our software package operator_gb...

```
sage: from operator_gb import *
sage: assumptions = [a*b*a - a,...]
sage: certify(assumptions, b - c)
```

Toy example: *"The Moore-Penrose inverse is unique"*

Recall: B is Moore-Penrose inverse of A if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

Claim If B and C satisfy these identities, then $B = C$

Proof Using our software package operator_gb...

```
sage: from operator_gb import *
sage: assumptions = [a*b*a - a,...]
sage: certify(assumptions, b - c)
b - c = (-c + c*a*c) + b*c_adj*(-a_adj + a_adj*b_adj*a_adj)
        - b*a*c*(-a*b + b_adj*a_adj) - b*(-a + a*c*a)*b
        + b*(-a*c + c_adj*a_adj) - b*(-a*c + c_adj*a_adj)*b_adj*a_adj
        - (-b + b*a*b) + (-c*a + a_adj*c_adj)*b*a*c
        - (-a_adj + a_adj*c_adj*a_adj)*b_adj*c + c*(-a + a*b*a)*c
        - (-b*a + a_adj*b_adj)*c + a_adj*c_adj*(-b*a + a_adj*b_adj)*c
```

Toy example: *"The Moore-Penrose inverse is unique"*

Recall: B is Moore-Penrose inverse of A if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

Claim If B and C satisfy these identities, then $B = C$

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        - (-a_adj + a_adj*c_adj*a_adj)*b_adj*c + c*(-a + a*b*a)*c
        - (-b*a + a_adj*b_adj)*c + a_adj*c_adj*(-b*a + a_adj*b_adj)*c
```

- Software produces cofactor representation (= algebraic proof)
- Statement is proven in all settings where linearity holds

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where

$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$

5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem

$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$

6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote

$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ ($m \ge n$) has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ ($m \le n$) has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A + E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank(A)}$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then

$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$

where

$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

In this paper we present new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. Some finite-dimensional results are extended to infinite-dimensional settings.

1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a) 1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} &\Leftrightarrow (A_1B_1)^\dagger(A_1A_1^* + A_2A_2^*) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow (A_1B_1)^\dagger A_2A_2^* = 0 \Leftrightarrow \mathcal{R}(A_2A_2^*) \subset \mathcal{N}((A_1B_1)^\dagger) \\ &\Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)^*) \Leftrightarrow B_1^*A_1^*A_2 = 0 \Leftrightarrow A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.

Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

“Every matrix has a Moore-Penrose inverse”

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Fact: A matrix $\quad\Rightarrow\quad \exists\, P, Q \;:\; PA^*A = A$ and $AA^*Q = A$

"Every matrix has a Moore-Penrose inverse"

Fact: A matrix $\Rightarrow$ $\exists\, P, Q \;:\; PA^*A = A$ and $AA^*Q = A$

Claim $\exists X : (PA^*A = A \wedge AA^*Q = A) \;\Rightarrow\; \mathsf{pinv}(A, X)$

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Strategy

1. Derive explicit expression for X
2. Plug in the explicit expression $\rightsquigarrow$ removes the existential quantifier
3. Prove by computing cofactor representations

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Proof Using our software package `operator_gb`...

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Proof Using our software package operator_gb...

```
sage: assumptions =  [a - p*a_adj*a,...]
sage: I = NCIdeal(assumptions + pinv(a,x))
sage: I.find_equivalent_expression(x)
```

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```
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sage: I = NCIdeal(assumptions + pinv(a,x))
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```

```
[- x + a_adj*q*x, - x + a_adj*p*x,
          - x + a_adj*q*p_adj, - x + a_adj*x_adj*x]
```

"Every matrix has a Moore-Penrose inverse"

Fact: A matrix $\Rightarrow$ $\exists\, P, Q \;:\; PA^*A = A$ and $AA^*Q = A$

Claim $\exists X : (PA^*A = A \wedge AA^*Q = A) \;\Rightarrow\; \mathrm{pinv}(A, X)$

Strategy

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Proof Using our software package operator_gb...

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```
[- x + a_adj*q*x, - x + a_adj*p*x,
        - x + a_adj*q*p_adj, - x + a_adj*x_adj*x]
```

$\Rightarrow$ $X = A^*QP^*$ is MP-inverse of A

(can be proven using the software)

Existential statements

In the previous example, we found a suitable expression.

Question Was this just luck?

Existential statements

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An existential statement is universally true if and only if explicit expressions exist and can be constructed as polynomial expressions in terms of the basic operators appearing in the statement.

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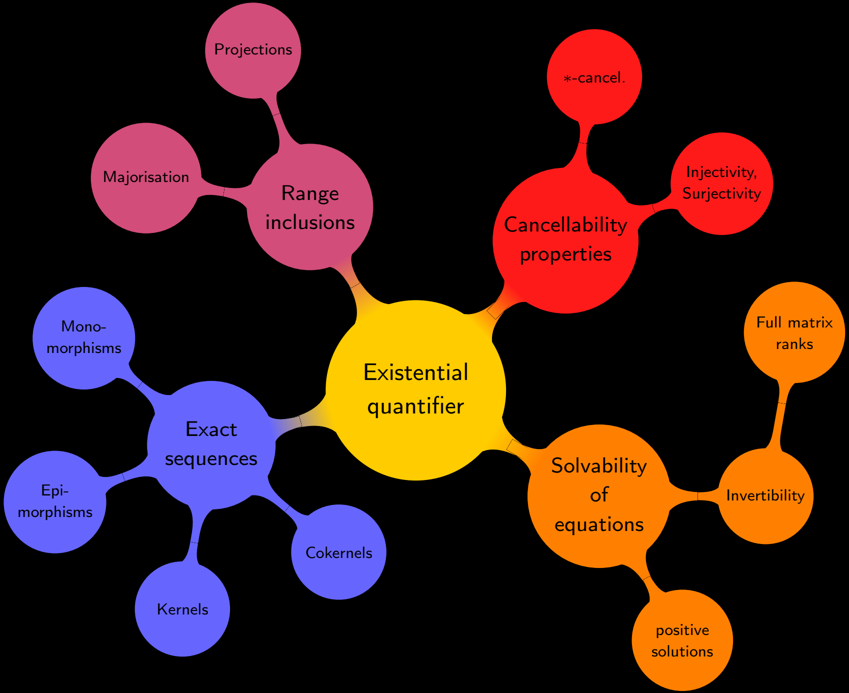
Reason Herbrand's theorem (Herbrand '30)

An existential statement is universally true if and only if explicit expressions exist and can be constructed as polynomial expressions in terms of the basic operators appearing in the statement.

- Enumerating all possible expressions is hopeless
- Requires good heuristics $\to$ provided by computer algebra
- Several heuristics implemented in `operator_gb` (ansatz, variable elimination, Gröbner basis techniques,...)

Existential quantifier


Existential quantifier
Range inclusions
Projections
Majorisation
Cancellability properties
$*$-cancel.
Injectivity, Surjectivity
Solvability of equations
Invertibility
Full matrix ranks
positive solutions
Exact sequences
Mono-morphisms
Epi-morphisms
Kernels
Cokernels

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m, n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where

$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$

5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem

$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$

6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \ne \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \ne \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote

$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \ne 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

Then

(a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.

(b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.

10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,

(a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;

(b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.

19. Let $A \in \mathbb{C}^{m\times n}$. Then

(a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.

(b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.

(c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.

(d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.

20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then

(a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.

(b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.

(c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.

(d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.

(e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.

(f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.

22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^* A_j = 0$, and $A_i A_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:

(a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.

(b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.

(c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.

(d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.

(e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.

26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A + E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m, n\}$. Suppose that A can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then

$$A^\dagger = \begin{bmatrix} A_{11}^* X A_{11}^* & A_{11}^* X A_{21}^* \\ A_{12}^* X A_{11}^* & A_{12}^* X A_{21}^* \end{bmatrix},$$

where

$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

In this paper we present new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. Some finite-dimensional results are extended to infinite-dimensional settings.

1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a) 1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{\;1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{\;1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.

Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

Applications

Handbook of Linear Algebra (20 ✔ / 6 ✔ / 4 ✖)

- yields statements with $\leqslant 70$ identities in $\leqslant 18$ basic operators
- cofactor representations consist of $\leqslant 226$ terms
- all proofs take $\sim$15 seconds altogether

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Triple reverse order law (Hartwig '86) A, B, C matrices.

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$$
$$\Longleftrightarrow$$
$$PQP = P, \qquad \mathcal{R}(A^*AP) = \mathcal{R}(Q^*), \qquad \mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$$

with $P = A^\dagger ABCC^\dagger, \quad Q = CC^\dagger B^\dagger A^\dagger A$

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Triple reverse order law (Milošević '19) A, B, C elements in C^*-algebra $\mathcal{R}$ with A, B, C, ABC MP-invertible.

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$$\Longleftrightarrow$$
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---

with $P = A^\dagger ABCC^\dagger, \quad Q = CC^\dagger B^\dagger A^\dagger A$

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Triple reverse order law (. . . '21) A, B, C elements in ring $\mathcal{R}$ with A, ~~B~~, C, ~~ABC~~ MP-invertible.

$$(ABC)^\dagger = C^\dagger \tilde{B}\, A^\dagger$$

$$\Longleftrightarrow$$

$$PQP = P, \qquad A^*AP\mathcal{R} \supseteq Q^*\mathcal{R}, \qquad CC^*P^*\mathcal{R} \subseteq Q\mathcal{R}$$

---

with $P = A^\dagger ABCC^\dagger$, $\quad Q = CC^\dagger \tilde{B}\, A^\dagger A$

Conclusion

Advantages

- Allows to automate lengthy computations
- Proofs are universal only requiring linearity
- Software allows to find minimal assumptions
- Software allows to find short proofs

Summary

- Framework for proving first-order statements about linear operators
- Correctness $\leftrightarrow$ existence of cofactor representations
- Approach is complete = Every true statement can be proven

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- Framework for proving first-order statements about linear operators
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What about your problems. . . ?