Gröbner bases in the free algebra: Introduction & advanced topics

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Séminaire Calcul Formel

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UNI KASSEL VERSITÄT

FWF Der Wissenschaftsfonds.

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- Ideal membership
- Elimination ideals
- Ideal/subalgebra intersections
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Studying finitely presented algebras

If $\mathcal{A} = K\langle X \mid R\rangle$, then Gröbner bases allow to

- decide whether $\mathcal{A}$ is trivial, commutative, finite dim.,...
- compute K-basis of $\mathcal{A}$
- decide word problem $f \stackrel{?}{=} g$ in $\mathcal{A}$

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Studying operator statements

- Model lin. operators by noncomm. polies
- Simplify and prove operator statements
- Validity of first-order operator statements
  $\Longleftrightarrow$
  nc ideal membership

(Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21, H, Raab, Regensburger '22)

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Algebraic setting

Free monoid $\langle X \rangle$ (on $X = \{x_1, \ldots, x_n\}$)

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Caution If $|X| > 1$, then $K\langle X \rangle$ is not Noetherian!

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Observe Since $\preceq$ is well-founded, $\to_G$ is terminating.

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Applications

K-basis: K-basis of $K\langle X \mid R\rangle$ is given by K-basis of $K\langle X\rangle/(R)$

Commutativity: $K\langle X \mid R\rangle$ is comm. iff $[x_i, x_j] \in (R)$ for all $i < j$

Algebraicity: $p \in K\langle X \mid R\rangle$ is alg. iff $(R + (p - y)) \cap K[y] \neq \emptyset$

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Well-behaved special cases

- $\dim_K(K\langle X\rangle/I) < \infty \Rightarrow$ every minimal GB of I is finite
- I homogeneous and finitely generated $\Rightarrow$ ideal membership decidable
- Many infinite GBs are finitely parametrisable $\Rightarrow$ ideal membership decidable
- Verifying ideal membership is always possible in finite time, and in practice this is often all we need.

Ambiguities & S-polynomials

Overlap ambiguity

$$f = \qquad + \cdots$$
$$g = \qquad + \cdots$$

$$f \qquad - \qquad g \quad \in \ \mathrm{SPol}(f, g)$$

Inclusion ambiguity

$$f = \qquad + \cdots$$
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Remarks

- Central part has to be non-trivial (coprime criterion)
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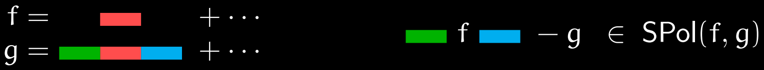

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- $xyxy$ has an (overlap) ambiguity with itself: $\begin{matrix} xyxy \\ xyxy \end{matrix}$

Buchberger's algorithm

Diamond lemma (Bergman '78)

$$G \subseteq K\langle X\rangle \text{ is GB of } (G) \quad \text{iff} \quad \forall \text{ S-poly } p \text{ of } G : p \xrightarrow{*}_G 0$$

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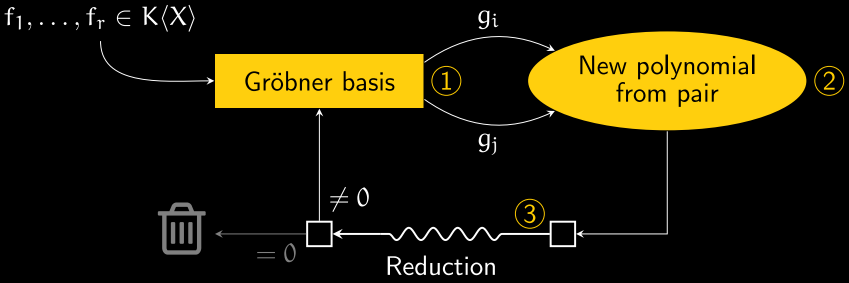

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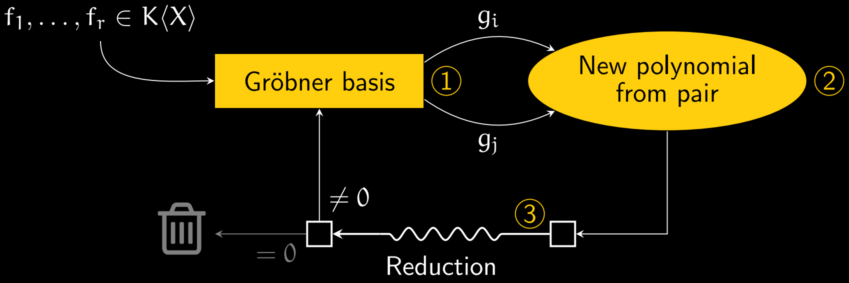

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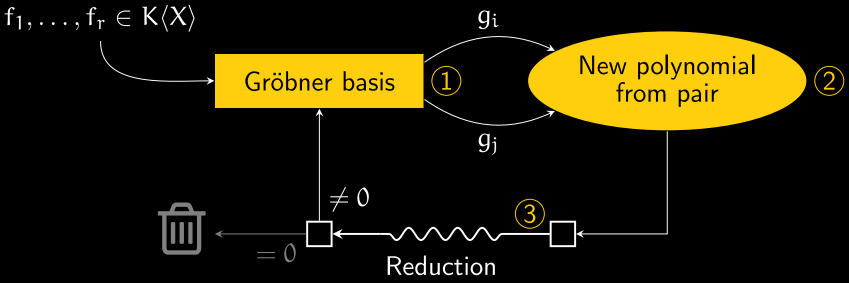

1. Selection: fair strategy *"Every S-poly is selected eventually"*
2. Construction: form S-polynomials from ambiguities
3. Reduction: reduction using (partial) Gröbner basis

Software

- Bergman: Gröbner bases in noncommutative algebras and in modules over them (Backelin et al. '06)
- Letterplace: SINGULAR package for noncommutative Gröbner bases (+ cofactor repr.) in free algebras, finitely presented algebras, and modules. Allows computations over $\mathbb{Z}$ (Levandovskyy, La Scala '09)
- Magma: Noncommutative F4 algorithm (Steel ~'09)
- NCAlgebra: MATHEMATICA package for simplification and reduction modulo noncommutative Gröbner bases (Helton, Stankus '01)
- GBNP: GAP package for noncommutative Gröbner bases for free and path algebras (Cohen, Gijsbers '03)
- OperatorGB: MATHEMATICA and SAGEMATH package for noncommutative Gröbner bases (+ cofactor repr.) and for automatically proving operator statements (H., Raab, Regensburger '19)
- SignatureGB (soon): SAGEMATH package for noncommutative signature Gröbner bases

Gröbner bases in the free algebra: Advanced topics

Hot topics

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Efficient computation

- $\rightsquigarrow$ Linear algebra reductions (Steel ~'09, Xiu '12)
- $\rightsquigarrow$ Signature-based algorithms (H., Verron '22)

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Expanding applicability

- $\rightsquigarrow$ Coefficient rings (Mikhalev, Zolotykh '98, Levandovskyy, Metzlaff, Abou Zeid '20)

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  1. Say we want to reduce $\{p_1, \dots, p_m\}$
  2. Find multiples of reducers needed for reductions (Symbolic preprocessing) $\rightsquigarrow$ $\{a_1 g_1 b_1, \dots, a_k g_k b_k\}$
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- Exploit efficient (sparse) linear algebra techniques and matrix structure

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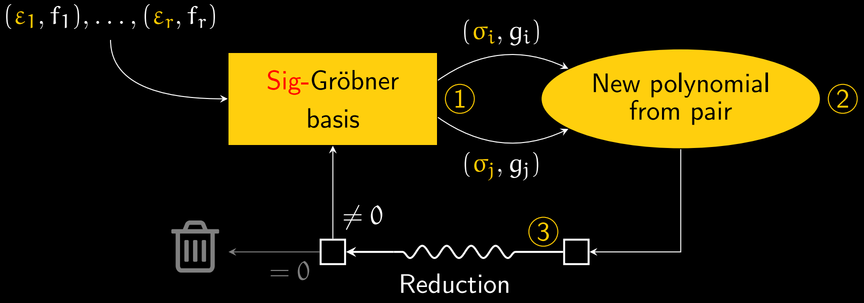
Regular operations

$$\sigma \succ \mu \Rightarrow (\sigma, f) \pm (\mu, g) =: (\sigma, f \pm g) \text{ (sig. preserved)}$$

$\Rightarrow$ regular reductions & S-polynomials

Buchberger's algorithm with signatures



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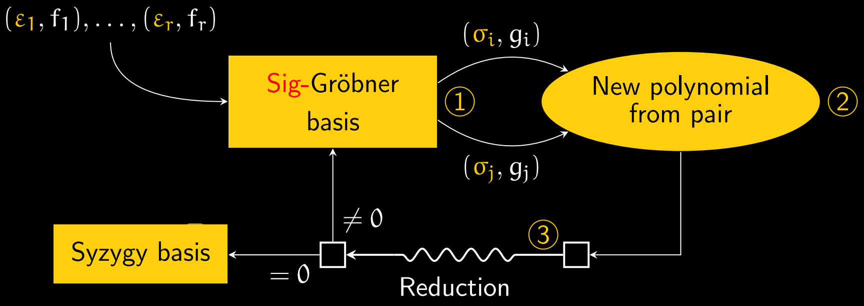
2. Construction: regular S-polynomials
3. Reduction: regular reductions using (partial) Sig-Gröbner basis

Buchberger's algorithm with signatures



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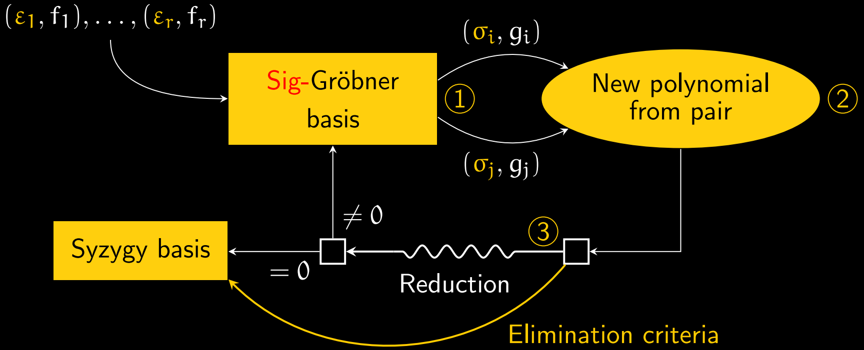

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Buchberger's algorithm with signatures

Output Signature Gröbner basis, allowing to recover

- a Gröbner basis of the ideal (+ cofactor representations)
- a Gröbner basis of the syzygy module

Remarks

- Termination is very rare – even less common than standard noncommutative GB algorithms
- Algorithm terminates iff ideal admits finite signature Gröbner basis
- Experimental data suggests performance improvement

Noncommutative Gröbner bases over rings

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$R\langle X \rangle \ldots$ free algebra over comm. PID (e.g. $R = \mathbb{Z}$)

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$$f \to_g f' \quad \Longleftrightarrow \quad \exists a, b \in \langle X\rangle : \mathrm{lm}(f) = \mathrm{lm}(agb) \;\&\; \mathrm{lc}(g) \mid \mathrm{lc}(f)$$

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Gröbner bases

Different notions, but most relevant are strong Gröbner bases.

Definition: $G \subseteq I$ s.t. $f \xrightarrow{*}_G 0$ for all $f \in I$

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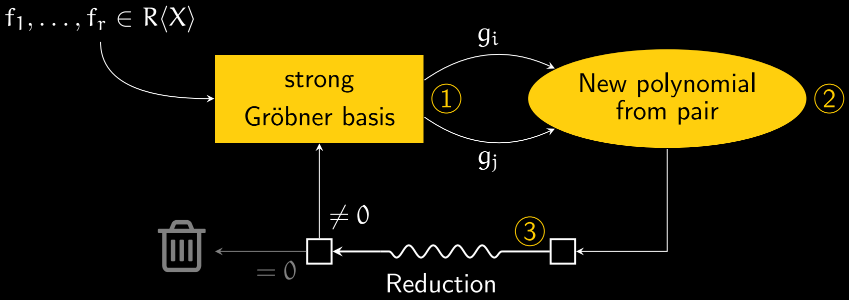
Problem $\mathrm{SPol}(f, g)$ and $\mathrm{GPol}(f, g)$ are infinite

$\Rightarrow$ can only compute up to some degree bound

Buchberger's algorithm over rings



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Conclusion

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- No termination guarantee $\rightsquigarrow$ Problems only semidecidable
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Advanced topics

- Linear algebra reductions $\rightsquigarrow$ Performance improvement
- Signature-based algorithms
  - Add module perspective to polynomials
  - Gröbner basis of ideal + syzygy module
  - Elimination criteria $\rightsquigarrow$ Performance improvement
- Gröbner bases over rings
  - Infinitely many S- & G-polynomials