Universal truth of operator statements via ideal membership

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UNIKASSEL VERSITÄT

FWF Der Wissenschaftsfonds.

Introduction

Goal Proving statements about matrices/linear operators automatically and efficiently!

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$$\begin{array}{c}\text{statement about}\\ \text{operators}\end{array}\Bigg\downarrow\begin{array}{c}\text{statement}\\ \text{is true}\end{array}$$

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$$\begin{tikzcd}
\text{statement about operators} \arrow[r, rightsquigarrow] \arrow[d] & \text{first-order formula} \\
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Modelling operator statements
via
many-sorted first-order logic

Many-sorted first-order logic

- First-order logic + sorts
- Same expressiveness as unsorted FO logic
- Computational advantages (sorts reduce # of expressions)
- We use sorts to model domains and codomains

Linear operator statements

“Statements”

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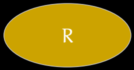

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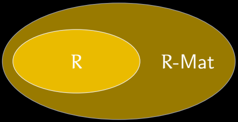

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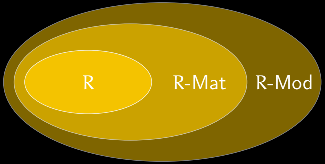

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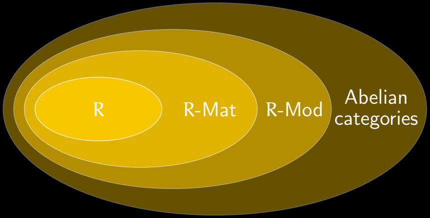

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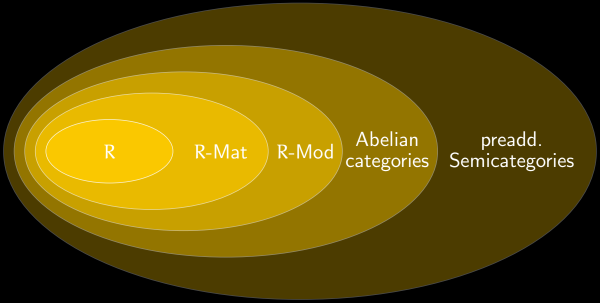

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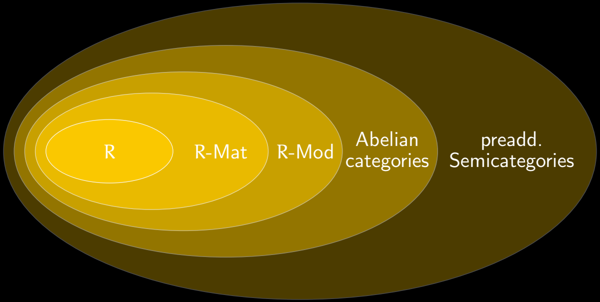

Semicategory = Category − identity morphisms

Signature

$$\mathbf{Var} = \{x_1, x_2, \dots\} \dots \text{variables}$$

A signature is a tuple $\Sigma = (O, C, F, \sigma)$ consisting of

- $O \dots$ object symbols $\qquad (u, v) \in O \times O \dots$ sort
- $C \dots$ constant symbols with $0_{u,v} \in C$ for $u, v \in O$;
- $F \dots$ function symbols with $-, +, \cdot \in F$;
- $\sigma \dots$ sort function assigning all symbols their sort

Syntax

Terms of sort (u, v)

- variables $x \in \mathbf{Var}$ with $\sigma(x) = (u, v)$
- constants $c \in C$ with $\sigma(c) = (u, v)$
- $f(t_1, \ldots t_n)$ with $\sigma(t_i) = (u_i, v_i)$ and

$$\sigma(f) = (u_1, v_1) \times \cdots \times (u_n, v_n) \to (u, v)$$

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Formulas

- $s \approx t$ with terms s, t where $\sigma(s) = \sigma(t)$
- $\neg\, \varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$
- $\exists\, x : \varphi$, $\forall\, x : \varphi$

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Fix $\quad \sigma(x) = (u, v), \qquad \sigma(y) = (u, w), \qquad \sigma(c) = (u, w)$

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$\forall x : x \approx 0 \cdot (c + c)$ arithmetic sentence

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$(c \not\approx 0 \wedge c + (-c) \approx 0) \rightarrow -c \not\approx 0$

arithmetic ground sentence

Axioms of semicategories

$$
\begin{aligned}
\mathcal{A}_{u,u',v,v'} = \{ & \\
& \forall x^{(v',v)}, y^{(u',v')}, z^{(u,u')} && : \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \\
& \forall x^{(u,v)}, y^{(u,v)}, z^{(u,v)} && : \quad x + (y + z) \approx (x + y) + z, \\
& \forall x^{(u,v)} && : \quad x + 0_{u,v} \approx x, \\
& \forall x^{(u,v)} && : \quad x + (-x) \approx 0_{u,v}, \\
& \forall x^{(u,v)}, y^{(u,v)} && : \quad x + y \approx y + x, \\
& \forall x^{(v',v)}, y^{(u,v')}, z^{(u,v')} && : \quad x \cdot (y + z) \approx x \cdot y + x \cdot z, \\
& \forall x^{(v',v)}, y^{(v',v)}, z^{(u,v')} && : \quad (x + y) \cdot z \approx x \cdot z + y \cdot z \\
\} &
\end{aligned}
$$

Axioms $\mathcal{A}$ of semicategories are then

$$\mathcal{A} = \bigcup_{u,u',v,v' \in O} \mathcal{A}_{u,u',v,v'}$$

Semantics

Formulas are true or false w.r.t. interpretation $\mathcal{I}$

Interpretations are as in classical FO logic, except that they respect the sorts.

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Universal truth of operator statement $\equiv$ $\mathcal{A} \models \varphi$

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Goal Prove semantic consequence via syntactic operations

Classically using some deductive system (e.g., Sequent calculus, Resolution, etc.)

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Theorem $\Psi \vDash \varphi$ iff $\Psi \vdash \varphi$

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Theorem

$$\begin{tikzcd} \Psi \vDash \varphi \arrow[rr, bend right, "\text{Completeness}"'] & \text{iff} & \Psi \vdash \varphi \arrow[ll, bend right, "\text{Correctness}"'] \end{tikzcd}$$

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Part II Derive analogous statement for $\Psi = \mathcal{A}$ with polynomial rhs

Expressing universal truth
by
polynomial ideal memberships

Interlude: Noncommutative polynomials

Noncommutative polynomials $=$ elements in free algebra $\mathbb{Z}\langle X\rangle$

$$= \sum_{i=1}^{d} c_i \cdot x_{i,1} \ldots x_{i,k_i}$$

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For $F \subseteq \mathbb{Z}\langle X\rangle$,

$$(F) = \left\{ \sum a_i f_i b_i \mid a_i, b_i \in \mathbb{Z}\langle X\rangle,\ f_i \in F \right\}$$

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Ideal membership problem $p \overset{?}{\in} (F)$ is semi-decidable

(e.g., using Gröbner bases)

From formulas to polynomials

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$$\exists x : a + b \approx x \quad \rightsquigarrow \quad ? \qquad \leftarrow \text{Herbrand's theorem}$$

$$f(a, b) \approx f(b, a) \quad \rightsquigarrow \quad ? \qquad \leftarrow \text{Ackermann's reduction}$$

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Corollary $\mathcal{A} \vDash \varphi$ iff $\mathcal{A} \vDash \varphi_1 \vee \cdots \vee \varphi_k$.

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1. Bind any free variables by universal quantifier
2. Move all quantifiers to the front
3. Apply the following rules exhaustively

$$\begin{aligned} \forall y : \psi \quad &\rightsquigarrow \quad \psi[y \mapsto c] \\ \exists x_1, \ldots, x_n \forall y : \psi \quad &\rightsquigarrow \quad \exists x_1, \ldots, x_n : \psi[y \mapsto f(x_1, \ldots, x_n)] \end{aligned}$$

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Proposition $\mathcal{A} \vDash \varphi \quad \text{iff} \quad \mathcal{A} \vDash \varphi^H$

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2. Replace every instance $f(t_1, \dots, t_n)$ by new constant $c_{f,t_1,\dots,t_n}$. Denote new formula by φ^{flat}.
3. For all $c_{f,s_1,\dots,s_n}$ and $c_{f,t_1,\dots,t_n}$ form functional consistency constraint

$$s_1 \approx t_1 \wedge \dots \wedge s_n \approx t_n \;\rightarrow\; c_{f,s_1,\dots,s_n} \approx c_{f,t_1,\dots,t_n}$$

4. $\varphi^{\text{FC}} \leftarrow$ Conjunction of all functional consistency constraints
5. $\varphi^{\text{Ack}} \leftarrow \left(\varphi^{\text{FC}} \rightarrow \varphi^{\text{flat}}\right)$

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Theorem φ valid iff φ^{Ack} valid

Corollary If non-arithmetic function symbol is removed, then

$$\mathcal{A} \vDash \varphi \qquad \text{iff} \qquad \mathcal{A} \vDash \varphi^{\mathsf{Ack}}$$

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Every φ is logically equivalent to a formula of the form

$$\mathrm{CNF}(\varphi) = \bigwedge_i \underbrace{\left(\bigvee_j s_{i,j} \not\approx t_{i,j} \vee \bigvee_k p_{i,k} \approx q_{i,k} \right)}_{\text{clause}}$$

Idealisation

Goal Translate $\mathcal{A} \vDash \varphi$ into polynomial predicate $I(\varphi)$

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clause:

$$I(C_i) \quad :\equiv \quad p_{i,k} - q_{i,k} \in (s_{i,1} - t_{i,1}, \ldots, s_{i,n} - t_{i,n}) \text{ for some } k$$

Idealisation

Goal Translate $\mathcal{A} \models \varphi$ into polynomial predicate $I(\varphi)$

Suffices to discuss arithmetic ground sentences

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Every φ is logically equivalent to a formula of the form

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Idealisation

clause:

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arith. ground sentence: $I(\varphi) \quad :\equiv \quad \bigwedge_{\substack{C \text{ clause} \\ \text{of } \mathrm{CNF}(\varphi)}} I(C)$

Main result

Theorem Let φ be an arithmetic ground sentence. Then

$$\mathcal{A} \vDash \varphi \qquad \text{iff} \qquad \mathrm{I}(\varphi) = \top.$$

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Advantages

- Axioms $\mathcal{A}$ are treated implicitly.
- Proof is independent of sorts, and thus, holds in all settings.
- Exploit efficient polynomial routines.

Semi-decision procedure

Input: signature Σ, formula φ

Output: $\top$ if and only if $\mathcal{A} \vDash \varphi$; otherwise infinite loop

1. $\varphi^{H} \leftarrow$ Herband normal form of φ
2. $\varphi_1, \varphi_2, \cdots \leftarrow$ an enumeration of $H(\varphi^{H})$
3. $n \leftarrow 1$
4. $\psi_n \leftarrow \bigvee_{i=1}^{n} \varphi_i$
5. $\psi_n^{\mathsf{Ack}} \leftarrow$ remove all non-arithmetic function symbols from ψ_n using Ackermann's reduction.
6. If $I(\psi_n^{\mathsf{Ack}}) = \top$, return $\top$. Otherwise, increase n by 1 and go to step 4.

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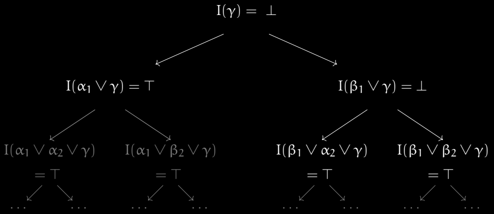
5. $\psi_n^{\mathsf{Ack}} \leftarrow$ remove all non-arithmetic function symbols from ψ_n using Ackermann's reduction.
6. For $k \leftarrow 1, \ldots, n$

   If $I(\psi_k^{\mathsf{Ack}}) = \top$ can be verified with n operations, return $\top$.
7. Increase n by 1 and go to step 4.

Computational aspects

- Efficiency of the procedure $\approx$ good enumeration of $\mathrm{H}(\varphi^{\mathrm{H}})$
  - Expert knowledge
  - "Polynomial unification"
- Avoid CNF blowup by incremental computation



- Treat simple subformulas separately

Applications

Used to automatically (im)prove statements in the field of

- generalized inverses
  - Moore-Penrose inverses (K. Bernauer)

Applications

(Extract from Handbook of Linear Algebra)

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where

$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$

---

10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,

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- Solvability of systems of equations

Applications

(Milošević, '20)

Theorem 2.2 *Let a_i, b_i, c_i be elements of a ring $\mathcal{R}$ with a unit such that a_i, b_i are regular and $a_i a_i^- c_i b_i^- b_i = c_i$ for $i = \overline{1,3}$. Additionally, let $s = a_2 l_{a_1}, j = a_3 l_{a_1}, m = j l_s, t = r_{b_1} b_2, k = r_{b_1} b_3, n = r_t k, p = a_3 l_{a_2}, q = r_{b_2} b_3$ and $s, j, m, t, k, n, r_m p, q l_n, r_{r_m p} r_m j, q l_k l_n \in \mathcal{R}^-$. The following are equivalent:*

(*i*) *The system of equations* (11) *is consistent.*

(*ii*) *The conditions*

$$
\begin{aligned}
&r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0\\
&r_{r_m p} r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- q l_n = r_{r_m p} r_m e l_n\\
&r_m j j^- e l_k l_n (q l_k l_n)^- q l_k l_n = r_m e l_k l_n
\end{aligned}
$$

are satisfied, where $e = c_3 - js^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)$.

(*iii*) *The conditions*

$$
\begin{aligned}
&r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0\\
&r_j(c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k = 0\\
&r_m(c_3 - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2) a_1^- c_1 b_1^- b_3) l_k l_n l_{q l_k l_n} = 0\\
&r_{r_{r_m p} r_m j} r_{r_m p} r_m (c_3 - a_3 a_2^- r_s c_2 t^- k - a_3 a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n = 0\\
&r_{r_m p} r_m (c_3 - js^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - js^- c_2 l_t b_2^- b_3\\
&-(a_3 - js^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n l_{q l_n} = 0.
\end{aligned}
$$

are satisfied.

In that case the general solution of (11) *is given by* (34), *where*

$$
\begin{aligned}
z_1 = {} & c_1 f + g^- r_s c_2 t^- t + a_1 l_{a_2} (r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e \\
& - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t \\
& + a_1 a_1^- l_g u_1 t t^- - a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t, \\
z_2 = {} & (c_1 b_1^- (b_3 - b_2 t^- k) + g^- r_s c_2 t^- k) l_n + a_1 a_3^- c_3 n^- n + a_1 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n \\
& a_1 l_{a_2} (r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
& - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] k^- k l_n + a_1 a_1^- l_g u_1 t^- k l_n \\
& - a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n, \\
z_3 = {} & g c_1 + s s^- c_2 l_t f^- + s (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 \\
& + s (l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
& + s s^- u_3 r_f b_1^- b_1 - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
& - s (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1, \\
z_4 = {} & g c_1 b_1^- (b_3 - b_2 t^- k) l_n + + s s^- c_2 l_t b_2^- b_3 l_n + (g g^- r_s c_2 + s s^- c_2) t^- k l_n + r_s a_2 a_3^- c_3 n^- n \\
& + r_s a_2 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n + s j^- r_m j [s^- a_2 a_3^- c_3 + (s^- a_2 (1 - a_3^- r_m r_j a_3) - j^- a_3) a_1^- u_2] n^- n \\
& + s (1 - j^- r_m j) s^- u_4 n^- n + s (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& + s (l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& + s s^- u_3 r_f b_1^- (b_3 - b_2 t^- k) l_n \\
& - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& - s (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n, \\
z_5 = {} & r_m ((a_3 - j s^- a_2) a_1^- c_1 + j s^- c_2 l_t f^-) + m m^- c_3 b_3^- b_1 + m m^- u_5 b_1^- (1 - b_3 l_k l_n b_3^-) b_1 \\
& + r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 + r_m j l_{r_{r_m p} r_m j} j^- r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 + r_m j s^- u_3 r_f b_1^- b_1 \\
& - r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
& - r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1, \\
z_6 = {} & r_m (a_3 - j s^- a_2) a_1^- c_1 f + r_m a_3 a_2^- r_s c_2 t^- t + r_m j s^- (c_2 l_t f^- f + c_2 t^- t) + m m^- c_3 b_3^- b_2 l_t \\
& + m m^- u_5 b_1^- (1 - b_3 l_k l_n b_3^-) b_2 l_t + m m^- [c_3 b_3^- b_2 t^- + u_5 b_1^- ((1 - b_3 l_k l_n b_3^-) b_2 t^- - b_3 k^-)] k l_n k^- t \\
& + m m^- u_6 t^- (1 - k l_n k^-) t + r_m (a_3 - j s^- a_2) a_1^- l_g u_1 t^- t \\
& + r_m (a_3 - j s^- a_2) a_1^- a_1 l_{a_2} (r_m p)^- r_m [e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
& - j (r_{r_m p} r_m j)^- r_{r_m p} r_m e - (1 - j (r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t \\
& - r_m (a_3 - j s^- a_2) a_1^- a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t,
\end{aligned}
$$

Applications

Used to automatically (im)prove statements in the field of

- Generalized inverses
  - Moore-Penrose inverses (K. Bernauer)
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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "j"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "j'"'] & E' \end{tikzcd}$$

Summary & Outlook

Summary

- Model operator statements via many-sorted logic
- Translate validity of operator statement into finitely many polynomial ideal memberships
- Tools: Herbrand's theorem + Ackermann's reduction

Outlook

- Producing proofs
- (Better) heuristics for finding good instantiations
- More advanced computational techniques (DPLL-style, techniques from SMT)
- Further applications