COMPUTING ELEMENTS OF CERTAIN FORM IN IDEALS TO PROVE PROPERTIES OF OPERATORS

Clemens Hofstadler, Clemens G. Raab, and Georg Regensburger

Johannes Kepler University, Linz, Austria

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JOHANNES KEPLER UNIVERSITY LINZ

Der Wissenschaftsfonds.

Motivation

Theorem (Arias, Gonzalez, 2010)

Let A, B, C be bounded linear operators on complex Hilbert spaces. Then,

$$\exists X : AXB = C \quad \Longleftrightarrow \quad \mathcal{R}(C) \subseteq \mathcal{R}(A) \text{ and } \mathcal{R}(C^*(A^\dagger)^*) \subseteq \mathcal{R}(B^*)$$

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Goal Prove such statements automatically!

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assumptions

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statement about operators $\xrightarrow{\text{translate}}$ noncommutative polynomials

noncommutative polynomials $\xrightarrow{\text{solve}}$ statement about polynomials

statement about polynomials $\xrightarrow{\text{apply framework}}$ assumptions $\Rightarrow$ claim

statement about operators $\xrightarrow{\text{formal computation}}$ assumptions $\Rightarrow$ claim

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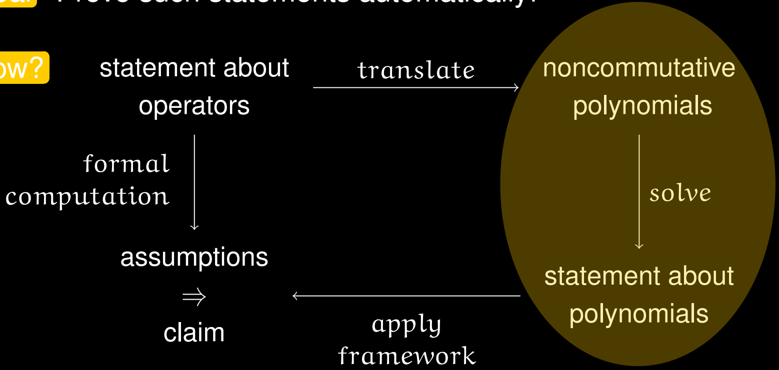

Algebraic setting

- $X = \{x_1, \ldots, x_n\}$ ... set of indeterminates
- $\langle X \rangle = \left\{ x_{i_1} \ldots x_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, n\} \right\}$ ... free monoid
- $K\langle X \rangle = \left\{ \sum_{m \in \langle X \rangle} c_m m \mid c_m \in K \right.$ such that $c_m = 0$ for almost all $\left. m \right\}$ ... free algebra in X over K
- For $F = \{f_1, \ldots, f_r\} \subseteq K\langle X \rangle$ we denote

$$(F) \quad = (f_1, \ldots, f_r) \quad = \left\{ \sum_i a_i f_{j_i} b_i \mid a_i, b_i \in K\langle X \rangle,\ f_{j_i} \in F \right\}$$

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Framework for verifying operator statements

(Raab, Regensburger, Hossein Poor, 2021)

Starting point: Statement about linear operators to prove

1. Translation:
   i. Phrase all properties in terms of identities
   ii. Convert identities into noncommutative polynomials
2. Solving: Find elements of certain form in ideal
3. Apply framework: General theorem guarantees existence of formal proof in terms of operators in all possible settings

Example

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Assumptions:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (A^\dagger)^* A^* = AA^\dagger, \quad A^*(A^\dagger)^* = A^\dagger A,$$

$$C = AY, \qquad C^*(A^\dagger)^* = B^* Z \qquad \text{+ adjoint statements}$$

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Claim: $\exists x : axb - c \in (f_1, \ldots, f_{12})$

Ideal intersections

Useful for finding polynomials f in ideal I of the form

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Given ideals $I, J \subseteq K\langle X \rangle$ and right ideals $I_\rho, J_\rho \subseteq K\langle X \rangle$.
Compute the intersections:

$\cap$	J_ρ	J
I_ρ	$I_\rho \cap J_\rho$	$I_\rho \cap J$
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I	$I = (G) = (G_\rho)_\rho$ $(G_\rho)_\rho \cap J_\rho$	$tI + (1-t)J + (tX - Xt)$ $\cap$ $K\langle X\rangle$

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$$az^*b - c \in (f_1, \dots, f_{12}) \cap (a, c)_\rho$$

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$$az^*b - c = f_1 y + (1 - aa^\dagger) f_5 + a f_{12}$$

Further techniques

How to find positive factorisations ?

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$$P \text{ positive } \iff \exists Q : P = Q^*Q$$

(in $K\langle X\rangle$: find $p - q^*q \in I$)

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(in $K\langle X\rangle$: find $p - q^*q \in I$)

Idea: Search for these elements by computing homogeneous polynomials in ideal.

Task: Given ideal $I \subseteq K\langle X\rangle$ and degree matrix $D \in \mathbb{Q}^{n\times m}$.
Compute $\mathrm{hom}_D(I) := (f \in I \mid f$ homogeneous w.r.t. $\deg_D)$.

Homogeneous part

Given: $I = (f_1, \ldots, f_r) \subseteq K\langle x_1, \ldots, x_n \rangle$

Task: Compute $\hom_{I_n}(I) = (f \in I \mid f$ homogeneous w.r.t. $\deg_{I_n})$

Generalisation of (Miller, 2016).

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$$\mathcal{A} = K\langle X, T, T^{-1} \rangle / (1 - t_i t_i^{-1}, x_j t_i - t_i x_j, t_i t_j - t_j t_i)$$

$$\varphi : K\langle X \rangle \to \mathcal{A}, \quad x_{i_1} \ldots x_{i_k} \mapsto [x_{i_1} \ldots x_{i_k} t_{i_1} \ldots t_{i_k}]$$

$$t.I = (\varphi(f_1), \ldots, \varphi(f_r))$$

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$$\mathrm{hom}_{I_n}(I) = t.I \cap K\langle X\rangle$$

Finding positive factorisations

Lemma (Arias, Gonzalez, 2010)

Let A, B, C be bounded linear operators such that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$. Then,

$$\exists X \text{ positive} : AXB = C \implies B^* A^\dagger C \text{ is positive}$$

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$\exists R : B^* A^\dagger C = R^* R$

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Proof with package `OperatorGB`[1]:

[1] Available at `https://clemenshofstadler.com/software/`

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Proof with package `OperatorGB`[1]:

```
In[1]:= (* Loading the package *)
SetDirectory[NotebookDirectory[]];
<< OperatorGB.m

Package OperatorGB version 1.4.1
Copyright 2019, Institute for Algebra, JKU
by Clemens Hofstadler, clemens.hofstadler@jku.at

In[3]:= (* Defining the assumptions and monomial ordering *)
assumptions = Join[{adj[b] - p ** a, a ** adj[y] ** y ** b - c, bac - adj[b] ** a† ** c}, Pinv[a]];
SetUpRing[{a, adj[a], a†, adj[a†], b, adj[b], c, adj[c], p, adj[p], y, adj[y], bac}]

(* Setting up the degree matrix *)
D = Table[0, {i, WordOrder // Length}, {j, 6}];
D[[1, 1]] = 1; D[[2, 1]] = -1;
D[[3, 2]] = 1; D[[4, 2]] = -1;
D[[5, 3]] = 1; D[[6, 3]] = -1;
D[[7, 4]] = 1; D[[8, 4]] = -1;
D[[9, 5]] = 1; D[[10, 5]] = -1;
D[[11, 6]] = 1; D[[12, 6]] = -1;
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Proof with package `OperatorGB`[1]:

```
In[12]:= (* Computing the reduced Gröbner basis *)
         G = Groebner[cofactors, assumptions];
         G = Interreduce[G][[1]];

In[15]:= (* Computing hom_D(I) *)
         hom = Hom[cofactors, G, 2, D];

         G has 694 elements in the beginning.

         Starting iteration 1...
         8416 ambiguities in total
         Iteration 1 finished. G has now 1269 elements

         Starting iteration 2...
         31810 ambiguities in total
         Iteration 2 finished. G has now 8715 elements

In[16]:= (* Found a positive factorisation *)
         hom[[26]]
Out[16]= -bac + adj[b] ** adj[y] ** y ** b
```

Monomial part

Given: $I_\rho = (f_1, \ldots, f_r)_\rho \subseteq K\langle x_1, \ldots, x_n \rangle$

Task: Compute $\mathrm{mon}(I_\rho) = (m \in I_\rho \mid m \text{ monomial})_\rho$

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$$\mathrm{mon}(I_\rho) = t.I_\rho \cap K\langle X\rangle$$

Approach does not immediately extend to two-sided ideals!

$$I = (x_1 - x_2) \quad \Longrightarrow \quad x_1 x_2 - x_2 x_1 \in t.I \cap K\langle X\rangle$$

Conclusion

Summary

- Techniques to compute elements of certain form in ideals
  - ideal intersections
  - homogeneous part
  - monomial part
  - ...
- Implemented in software package `OperatorGB`
- Used to automatically prove statements about operators (e.g. generalized inverses, homological algebra, ...)

Outlook

- Investigate possibilities for optimisation
- Ideal theoretic operations for further classes of elements