AUTOMATIZING PROOFS OF PROPERTIES OF OPERATORS

Clemens Hofstadler, Clemens G. Raab, and Georg Regensburger

Johannes Kepler University, Linz, Austria

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JOHANNES KEPLER UNIVERSITY LINZ

FWF Der Wissenschaftsfonds.

Motivation

Theorem (Werner, 1994)

Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$ with inner inverses A^- and B^-. If A^-ABB^- is idempotent, then B^-A^- is an inner inverse of AB.

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noncommutative polynomials $\xrightarrow{\text{solve}}$ statement about polynomials

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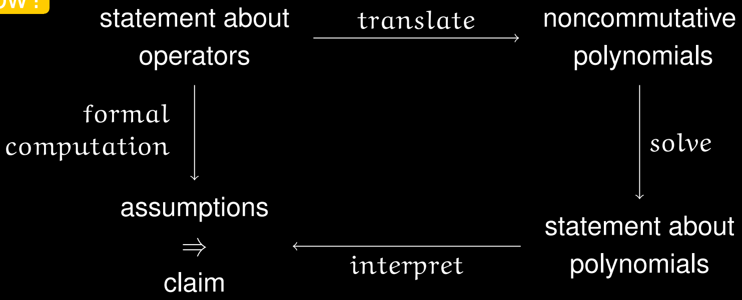

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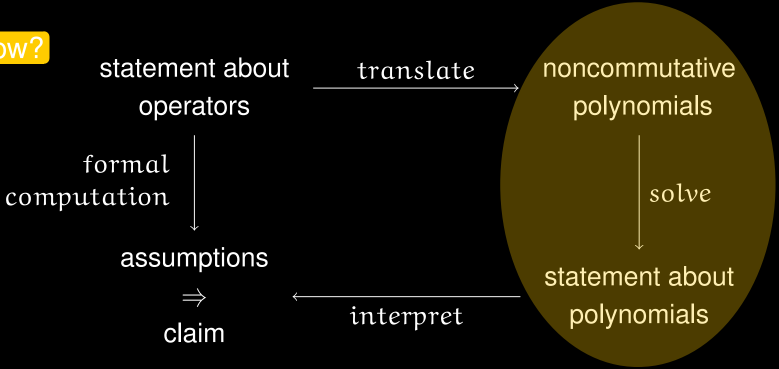

Framework for verifying operator statements

(Raab, Regensburger, Hossein Poor, 2021)

Starting point: Statement about matrices or operators to prove

1. Translation:
    1. Phrase all properties in terms of identities
    2. Convert identities into noncommutative polynomials
2. Solving: Verify ideal membership of claim
3. Interpretation: Consider different settings

More formally...

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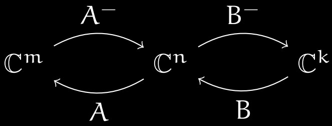

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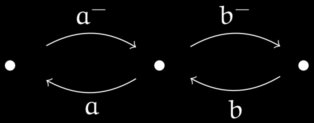

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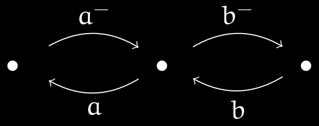

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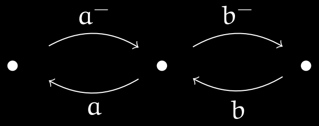

Quiver representations in an $\mathbb{R}$-linear category

Vertices and edges of Q are assigned objects and morphisms.

$f \in R\langle X \rangle$ compatible with $Q \iff$ monomials of f are paths in Q with same start and end.

$\Rightarrow$ Compatible polynomials have realizations as morphisms.

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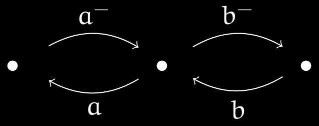

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Theorem (Raab, Regensburger, Hossein Poor, 2021)

Let $F \subseteq R\langle X\rangle$ and $f \in (F)$. Then, for every labelled quiver Q and every representation of Q in an R-linear category s.t.

1. f and all elements of F are compatible with Q, and
2. realizations of all elements of F are zero,

we have that the realization of f is zero.

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Assumptions:

$$AA^-A = A, \qquad BB^-B = B, \qquad (A^-ABB^-)^2 = A^-ABB^-$$

Claim:

$$ABB^-A^-AB = AB$$

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$$aa^-a - a, \qquad bb^-b - b, \qquad (a^-abb^-)^2 - a^-abb^-$$

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Then, "assumptions $\Rightarrow$ claim" since

$$f = f_1(bb^-b - bb^-a^-abb^-b) + (a - abb^-a^-a)f_2 + af_3b.$$

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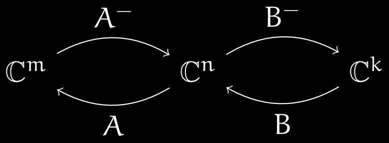

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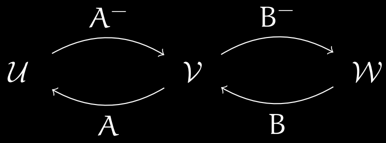

Applications

Framework implemented in the MATHEMATICA and SAGEMATH package `OperatorGB`. Available at

`https://clemenshofstadler.com/software/`

Successfully used to automatically (im)prove statements in the field of

- generalised inverses (more specifically: reverse order laws)
- homological algebra (more specifically: diagram chases)

Example: product of three Moore-Penrose inverses

Theorem (Hartwig, 1986)

A, B, C matrices s.t. $M = ABC$ exists. Let $P = A^{\dagger}ABCC^{\dagger}$ and $Q = CC^{\dagger}B^{\dagger}A^{\dagger}A$.
Then, $PQ = (PQ)^2, \mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$ iff

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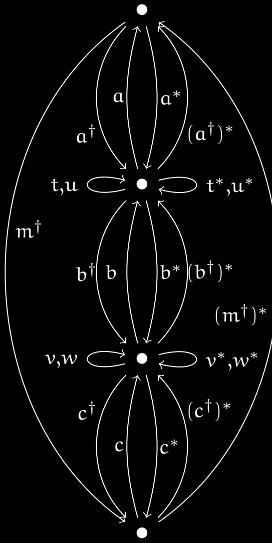

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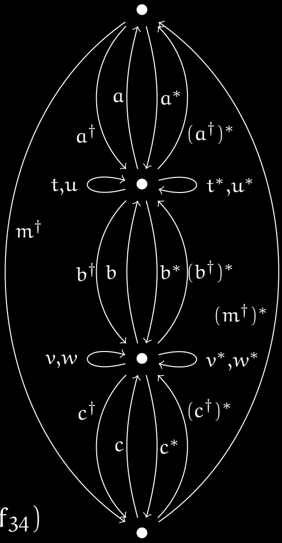

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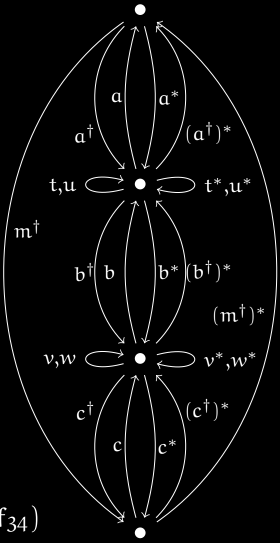

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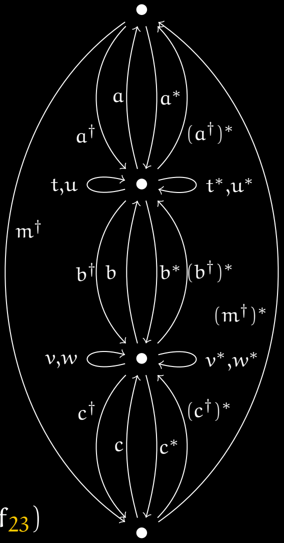

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Consider the following commutative diagram with exact rows in an abelian category.

If α is an epimorphism,
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then γ is an isomorphism.

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Demonstration in MATHEMATICA

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  - Proofs rely on Gröbner bases and reduction to zero
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Outlook

- Integration of further inference steps + theoretical foundation
- Find further areas of application