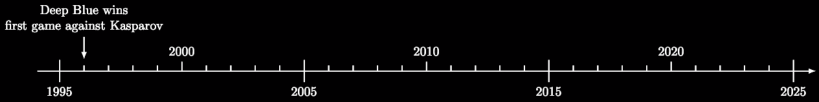
SOLVING QBFS WITH ALPHAZERO

Clemens Hofstadler, supervised by Univ.-Prof. Dr. Martina Seidl
Institute for Algebra, JKU Linz
Seminar Algebra and Discrete Mathematics, 10 June 2021

JOHANNES KEPLER
UNIVERSITY LINZ

THE ALPHAZERO FRAMEWORK

History


Deep Blue wins
first game against Kasparov
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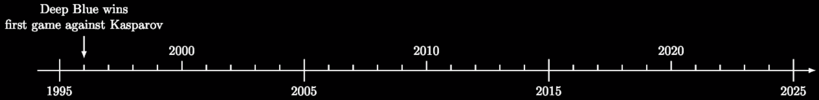
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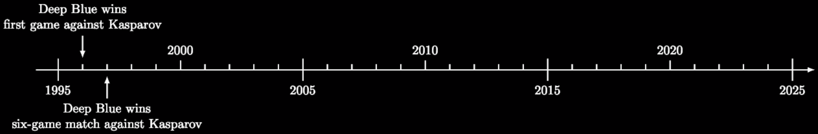
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six-game match against Kasparov

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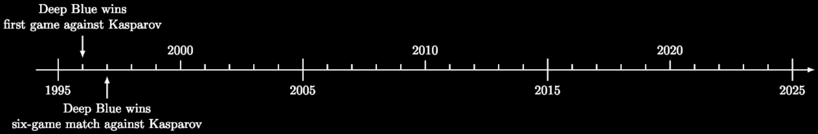
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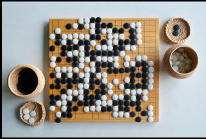

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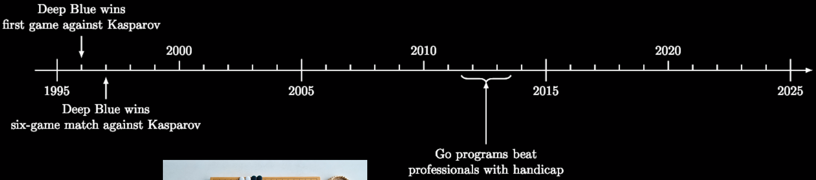
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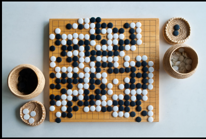

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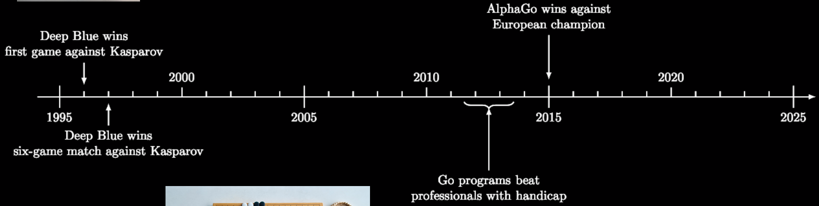
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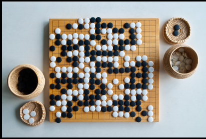

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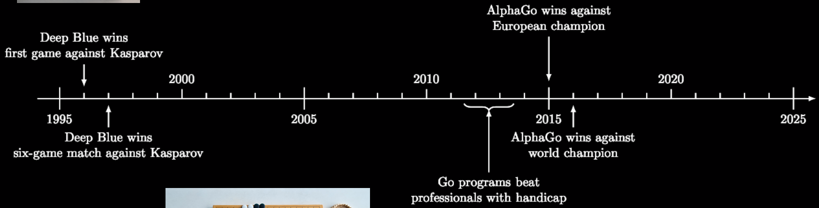
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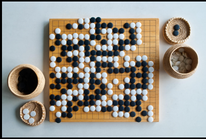
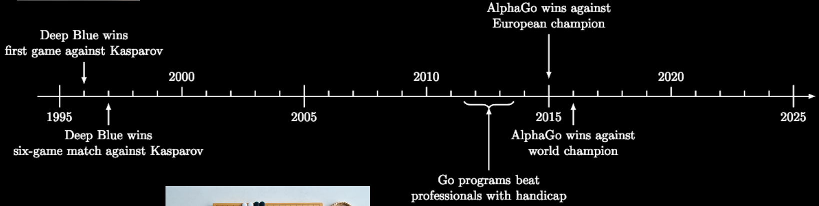


History


NETFLIX
ALPHAGO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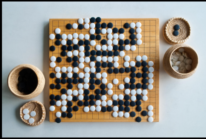

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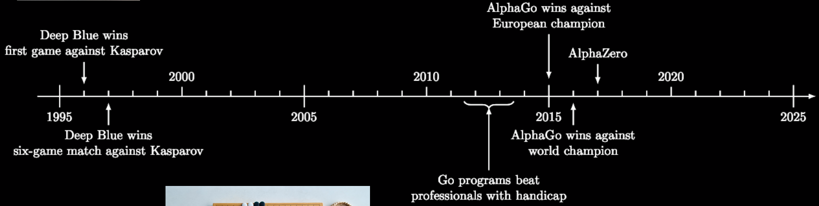
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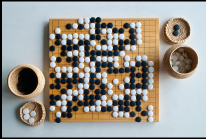

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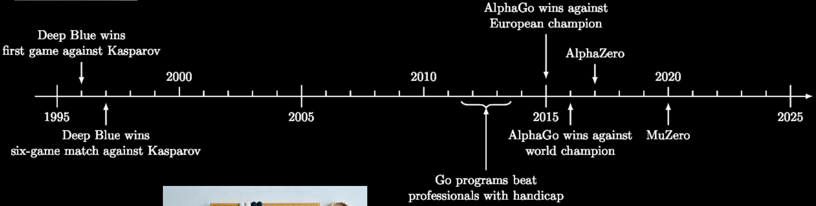
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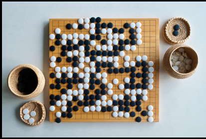

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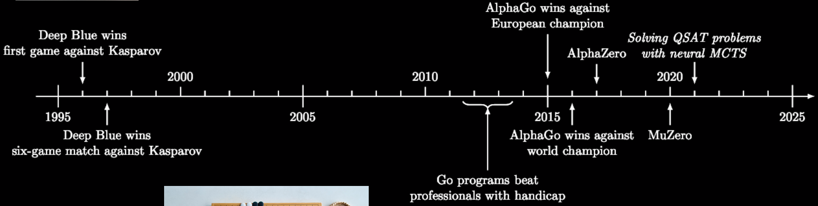
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Solving QSAT problems
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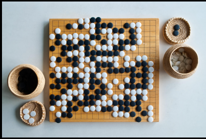

AlphaZero

- AI framework to learn two-player, fully-observable, symmetric games
- AlphaZero learns (almost) tabula-rasa

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AlphaZero =

Monte Carlo tree search + Self-play reinforcement learning

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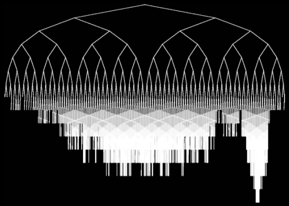
$$\text{AlphaZero} = \text{Monte Carlo tree search} + \text{Self-play reinforcement learning}$$



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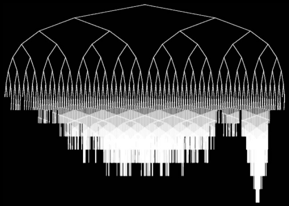

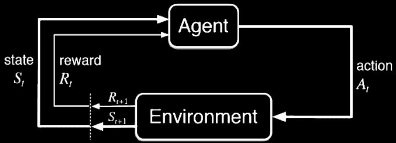

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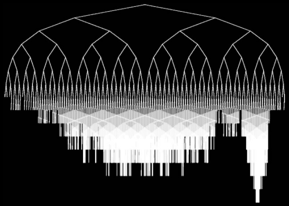

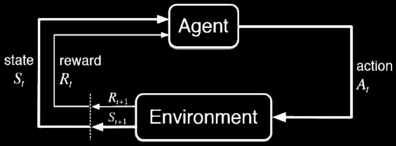

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AlphaZero $=$

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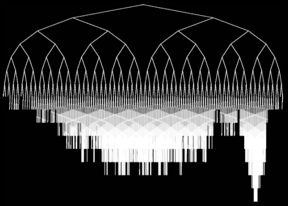
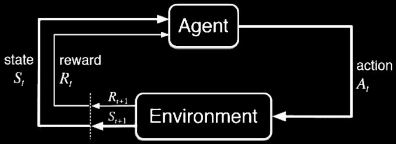




vs.

AlphaZero

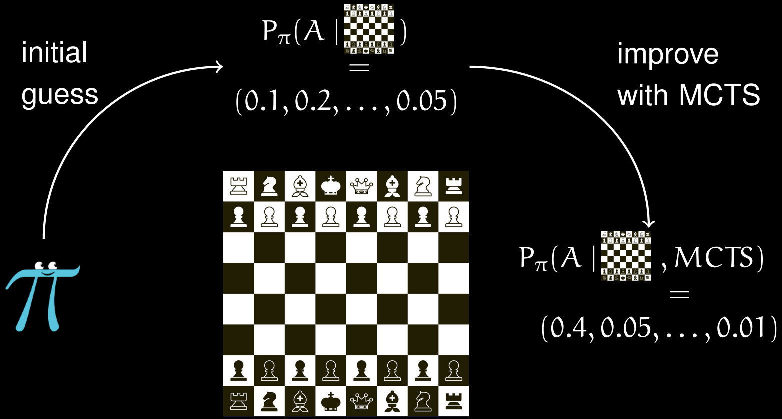
AlphaZero

initial guess

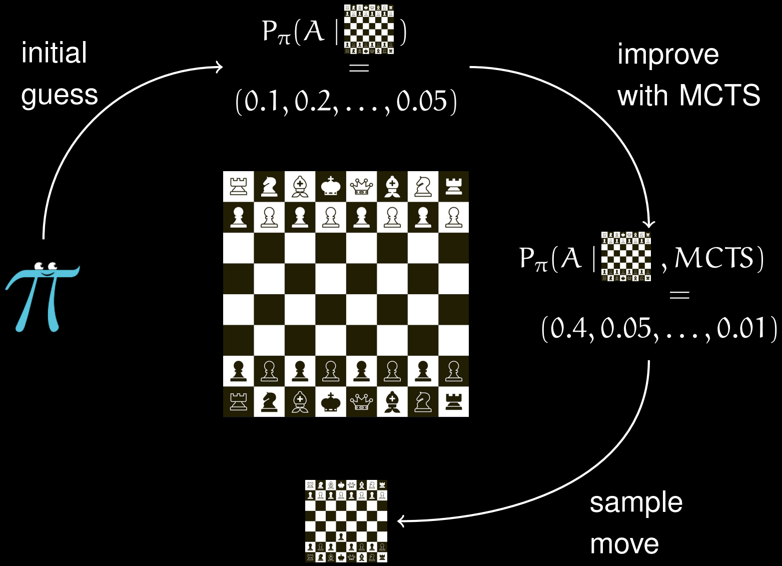
$P_\pi(A \mid$  $)$

$$= (0.1, 0.2, \ldots, 0.05)$$

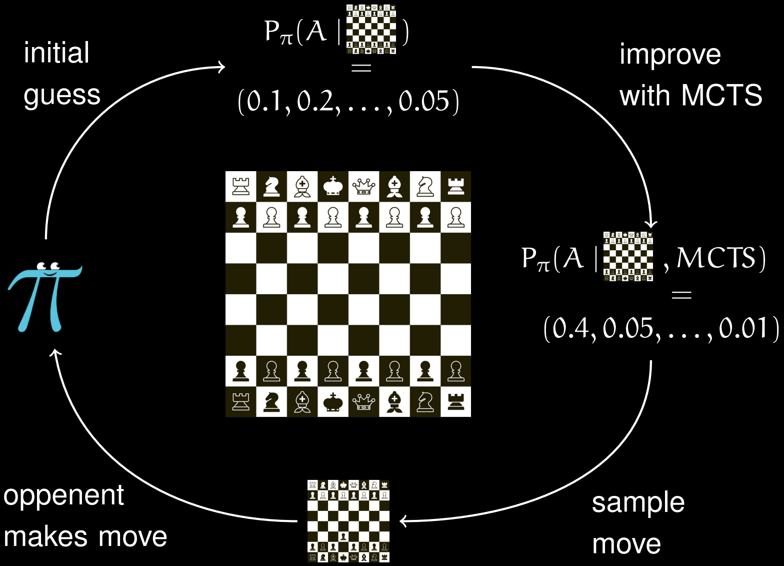
AlphaZero


initial guess
$P_\pi(A \mid \cdot) = (0.1, 0.2, \ldots, 0.05)$
improve with MCTS
$P_\pi(A \mid \cdot, MCTS) = (0.4, 0.05, \ldots, 0.01)$

AlphaZero


initial guess
$P_\pi(A \mid \text{[board]}) = (0.1, 0.2, \ldots, 0.05)$
improve with MCTS
$P_\pi(A \mid \text{[board]}, \text{MCTS}) = (0.4, 0.05, \ldots, 0.01)$
sample move

AlphaZero



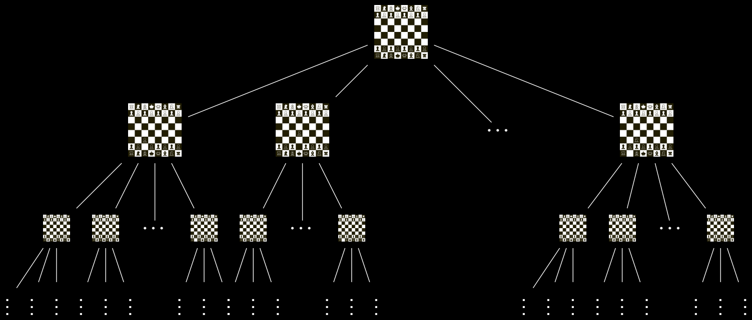
$P_\pi(A \mid \text{board}) = (0.1, 0.2, \ldots, 0.05)$
initial guess
improve with MCTS
$P_\pi(A \mid \text{board}, MCTS) = (0.4, 0.05, \ldots, 0.01)$
sample move
oppenent makes move

MCTS

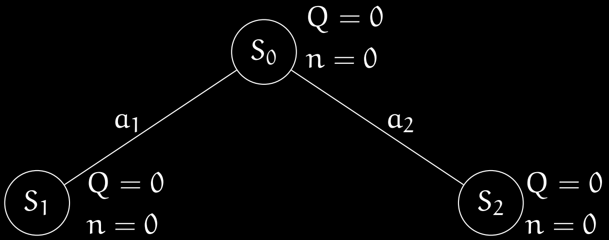
Given: Two-player, fully-observable game with game tree

Task: Find promising action in current state

Example:

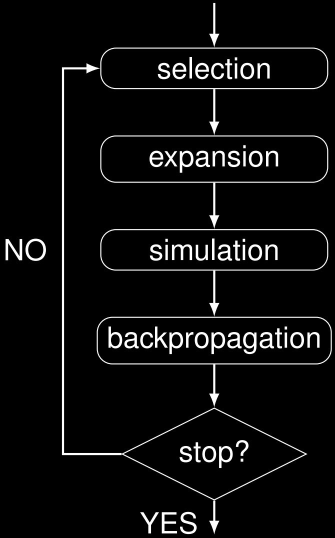


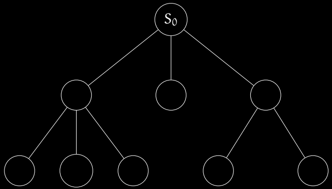
MCTS



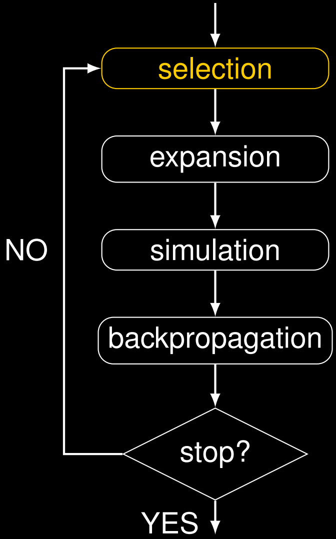
- States: S_0, S_1, S_2
- Actions: a_1, a_2
- Attributes of nodes
  - Q: current value of the node
  - n: number of visits

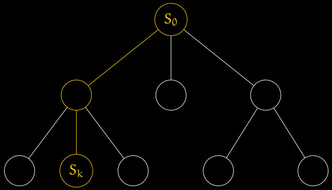
MCTS


selection
expansion
simulation
backpropagation
stop?
NO
YES


s_0

MCTS





Traverse tree following
upper confidence bound

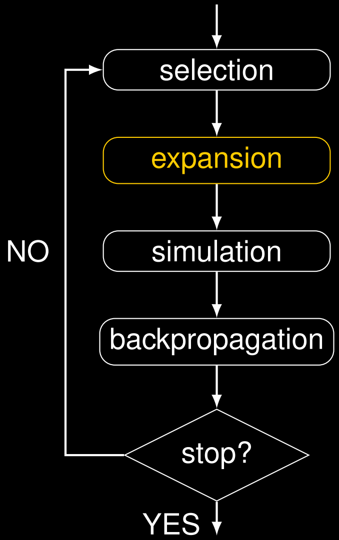
$$\text{UCB}(S) = \frac{Q}{n} + \gamma\sqrt{\frac{\ln N}{n}}$$

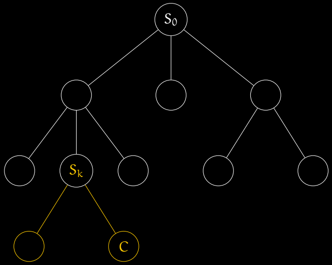
$$\text{UCB}(S) = \infty, \quad \text{if } n = 0.$$

γ . . . hyperparamter

N . . . # visits of parent node

MCTS





if not visited S_k and $k \neq 0$

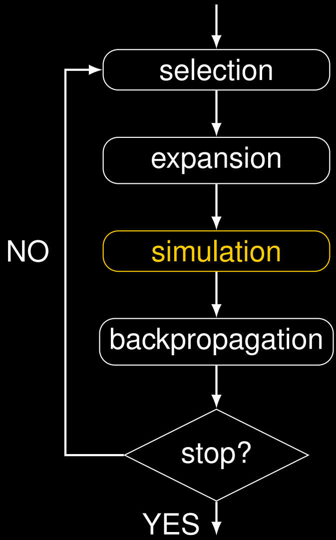
simulate S_k

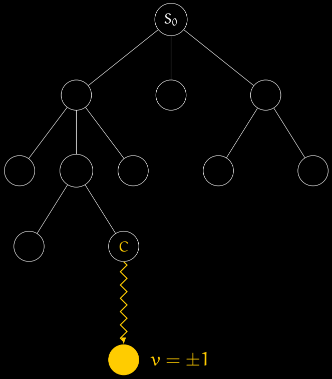
else

expand S_k

simulate child C

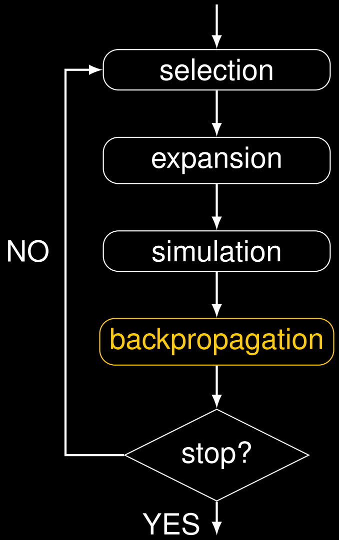
MCTS

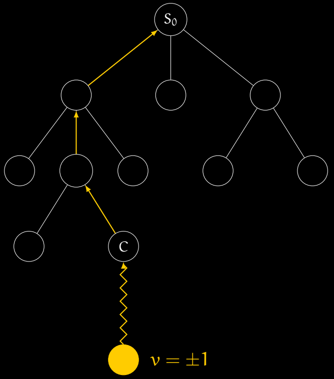




Play random game.

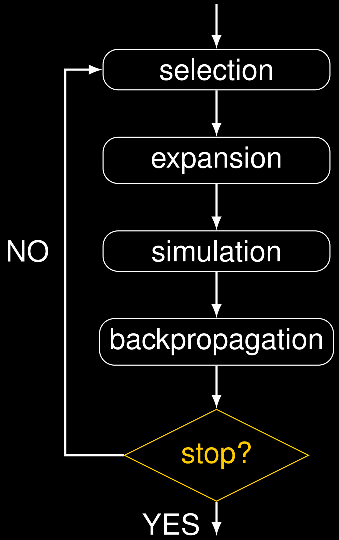
MCTS

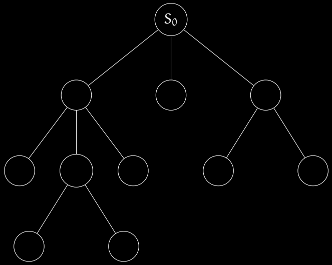




Update Q values for nodes on path from C to S_0 using v.

MCTS



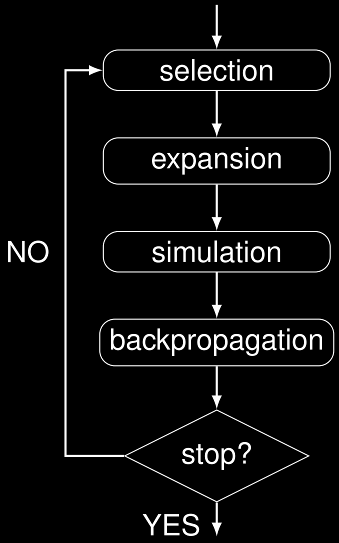


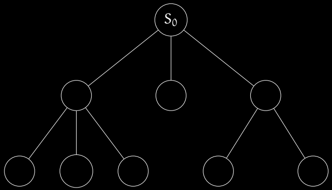
Termination criteria:

- Timeout
- Max. number of iterations exceeded

Return relative frequencies

MCTS in AlphaZero





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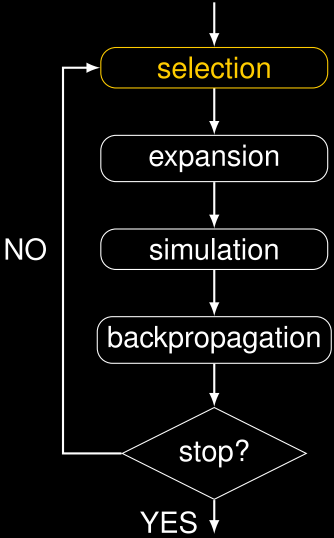
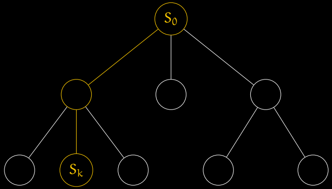




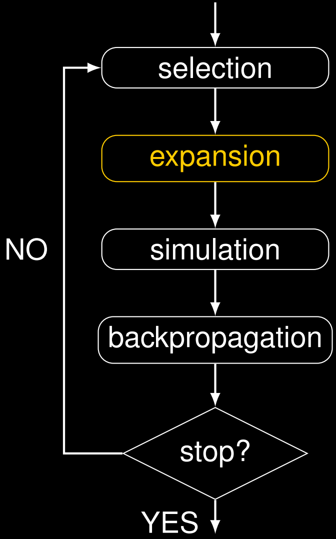
Traverse tree following
Predictor + UCB

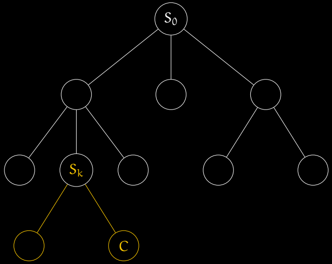
$$\text{PUCB}(S) = \frac{Q}{n+1} + \gamma \frac{P_\pi(a \mid P(S))\sqrt{N}}{n+1}$$

$P(S)$ ... Parent of S

a ... action from $P(S)$ to S

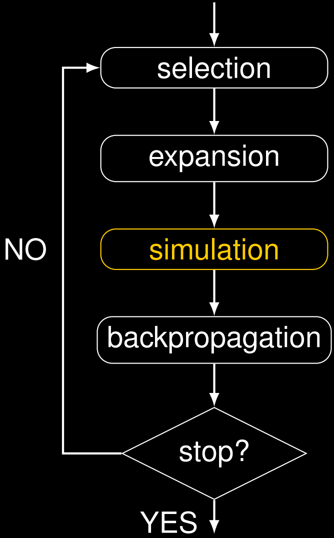
MCTS in AlphaZero

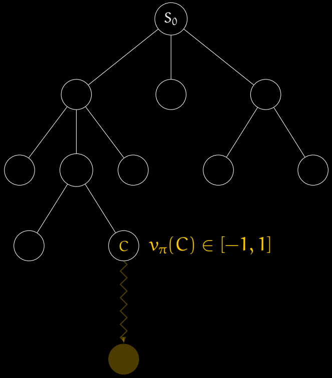




if not visited S_k
  compute $P_\pi(A \mid S_k) = p_\pi(S_k)$
if not visited S_k and $k \neq 0$
  simulate S_k
else
  expand S_k
  simulate child C with action
  $a = \text{argmax}_{a'} P_\pi(a' \mid S_k)$

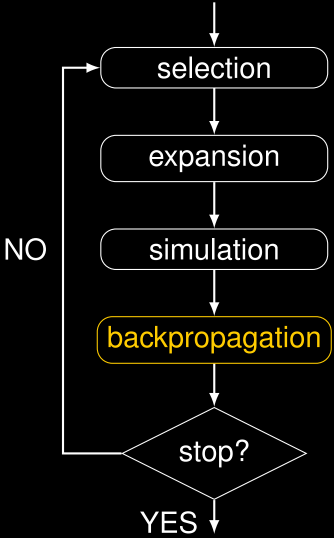
MCTS in AlphaZero

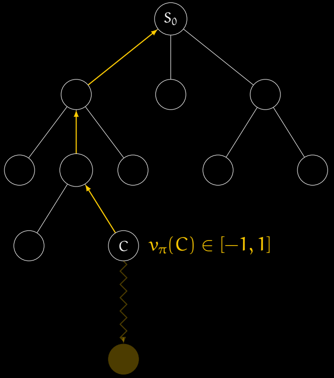




Estimate $v \approx v_\pi(C)$ using the neural network.

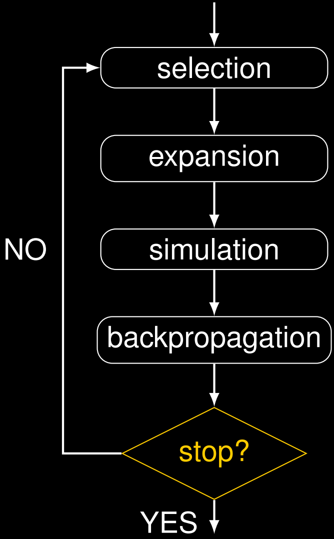
MCTS in AlphaZero

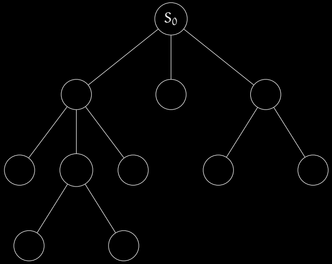




Update Q values for nodes on path from C to S_0 using $v_\pi(C)$.

MCTS in AlphaZero





Termination criteria:

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Return relative frequencies

Training by self-play

```
Initialize player with random parameters π
for e ← 1, . . . , E do
    Generate data by self-play games
    Update parameters π
    Compare new player to best player
return π
```

Training by self-play

3: Generate data by self-play games

$t \leftarrow 1$
for $k \leftarrow 1, \ldots, N$ **do**
  $S_t \leftarrow$ initial board
  while S_t is not an end state **do**
    $p_t \leftarrow P_\pi(A \mid S_t, MCTS)$
    sample move a_t from p_t
    save data (S_t, p_t, z_t), where z_t is the game result
    $S_{t+1} \leftarrow$ new board after doing move a_t
    $t \leftarrow t + 1$
return $\{(S_t, p_t, z_t) \mid 1 \leq t \leq T\}$

Training by self-play

4: Update parameters π

1: Given data $\{(S_t, p_t, z_t) \mid 1 \leq t \leq T\}$, use gradient descent to update parameters π to minimize

$$L = \sum_{t=1}^{T} \left((z_t - v_\pi(S_t))^2 - p_t^T \log p_\pi(S_t) \right) + c\|\pi\|^2,$$

with hyperparameter c.

Training by self-play

5: Compare new player to best player

$\pi' \leftarrow$ parameters of the best previous player
let π play M games against π'
if π wins $\geq 55\%$ of these game **then**
    mark π as the best player
else
    $\pi \leftarrow \pi'$
return π

ALPHAZERO FOR QBF SOLVING

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QBF = Quantified Boolean Formula

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- Canonical PSPACE-complete problem
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clause: $(v \vee \bar{y} \vee \bar{z})$

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- $\mathcal{Q}.\varphi$ is true iff existential player has winning strategy
- $\mathcal{Q}.\varphi$ is false iff universal player has winning strategy

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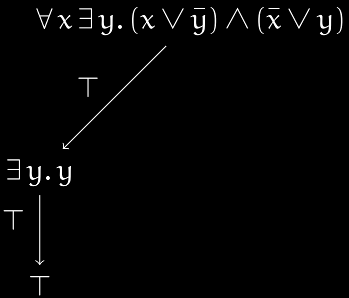
QBF solving as a game



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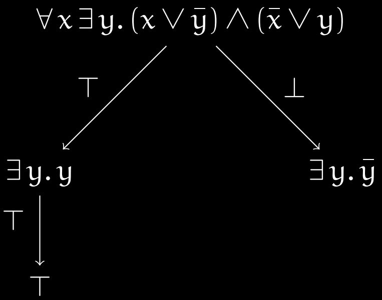

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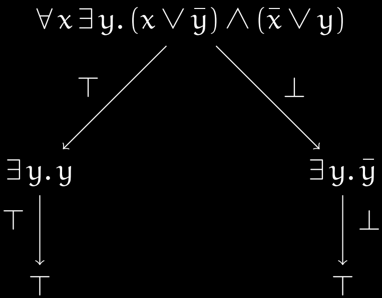

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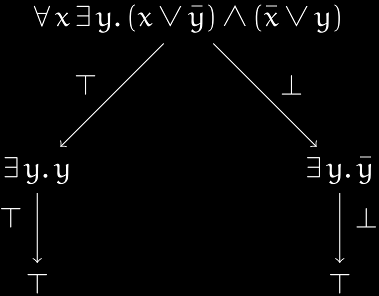

Given two perfect players, one game suffices to solve a QBF.

Solving QBFs with AlphaZero

Idea:

1. Use AlphaZero to learn two (perfect) players $\pi_\exists$, $\pi_\forall$.
2. Given $\mathcal{Q}.\varphi$, let $\pi_\exists$ play against $\pi_\forall$.
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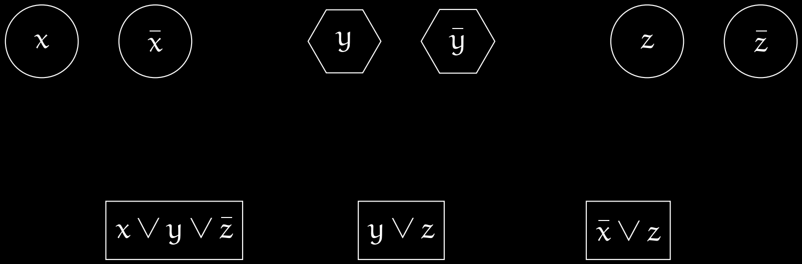

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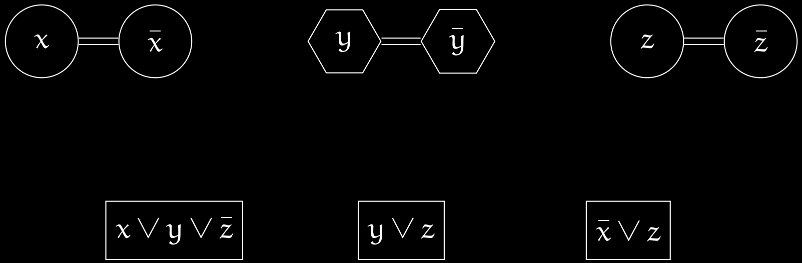

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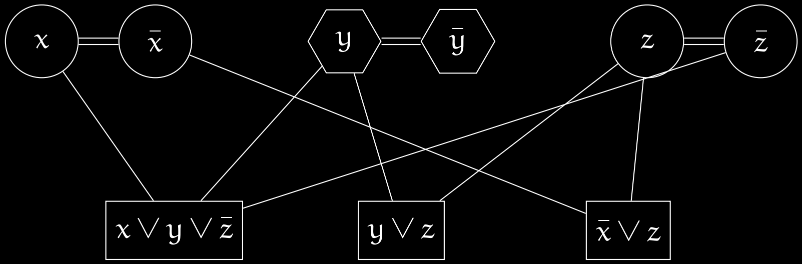

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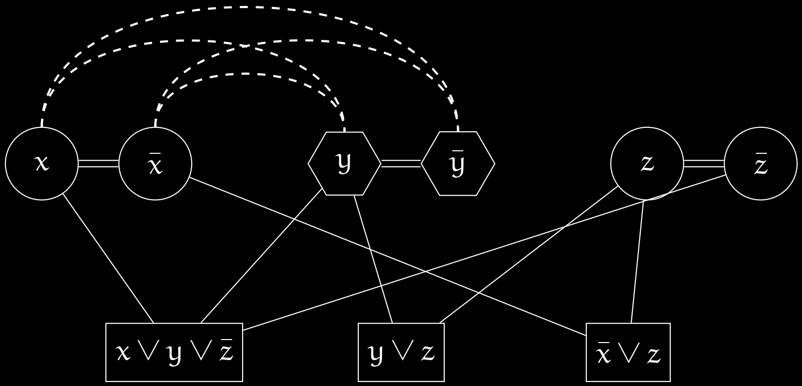

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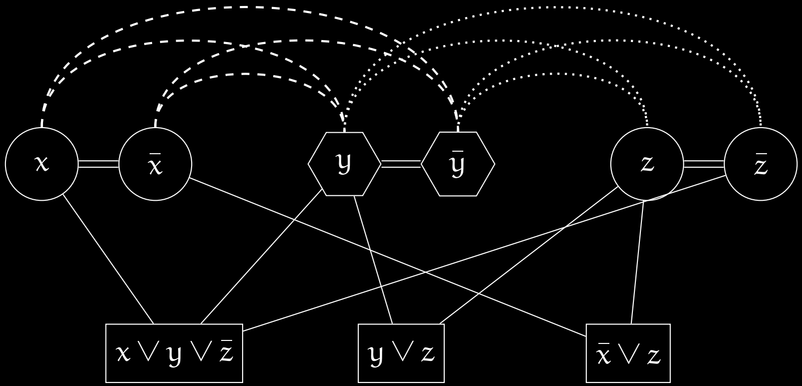

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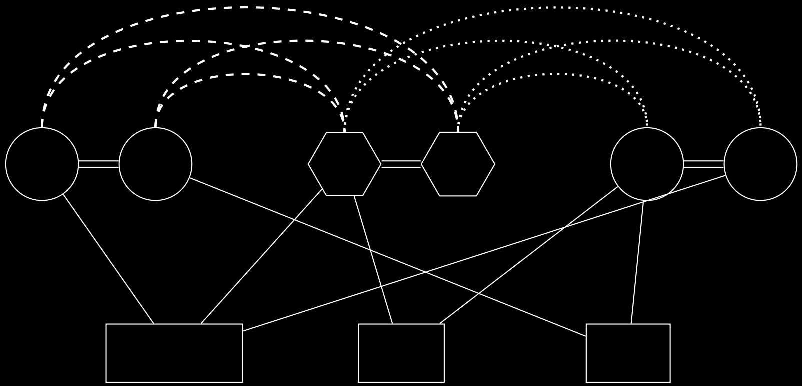

How to represent QBFs?

Process QBF graphs with a gated graph neural network.

1: Initialize $h_v^{(0)}$ for each vertex v with a vector in $\mathbb{R}^N$

2: **for** $t \leftarrow 0, \ldots, T-1$ **do**

3: $\quad m_v^{(t+1)} \leftarrow \sum_{w \in N(v)} A_{e_{vw}} h_w^{(t)}$ for each vertex v

4: $\quad h_v^{(t+1)} \leftarrow \mathrm{GRU}(h_v^{(t)}, m_v^{(t+1)})$ for each vertex v

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Solution:

- Extend AlphaZero to train two networks with different goals simultaneously
- Take more care during the MCTS

Training by 2-player-self-play

Initialize players with random parameters $\pi_\exists, \pi_\forall$

for $e \leftarrow 1, \ldots, E$ **do**

    Generate data by self-play games

    Update parameters $\pi_\exists$

    Compare new player to best existential player

    Generate data by self-play games

    Update parameters $\pi_\forall$

    Compare new player to best universal player

return $\pi_\exists, \pi_\forall$

EXPERIMENTAL RESULTS

Training

Starting point

`https://github.com/suragnair/alpha-zero-general`

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on a training set consisting of

- 100 random QBFs (50 true, 50 false) with
- 10 – 40 variables and
- 5 – 20 clauses

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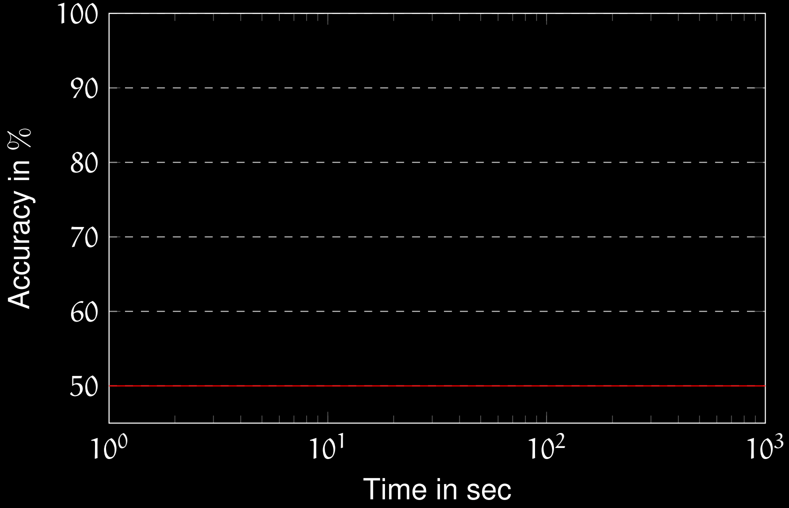

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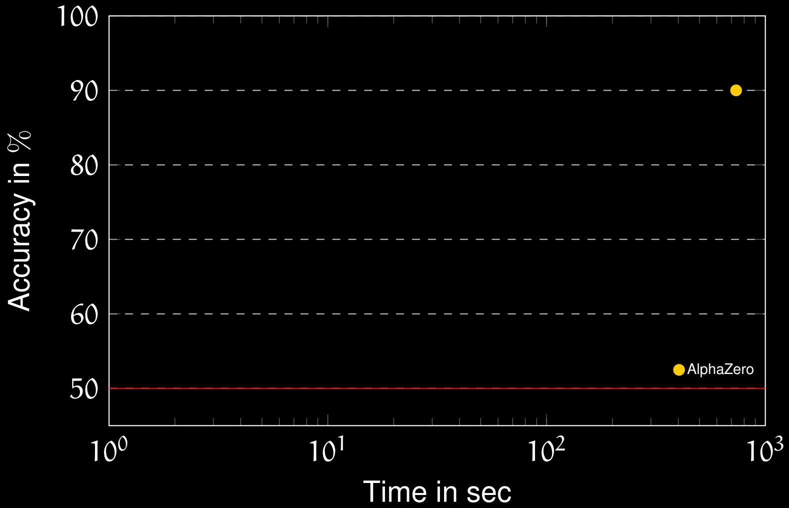

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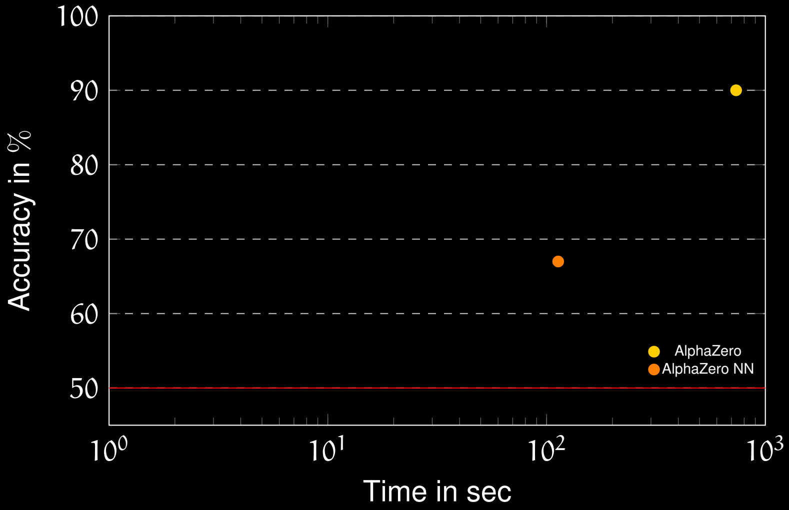

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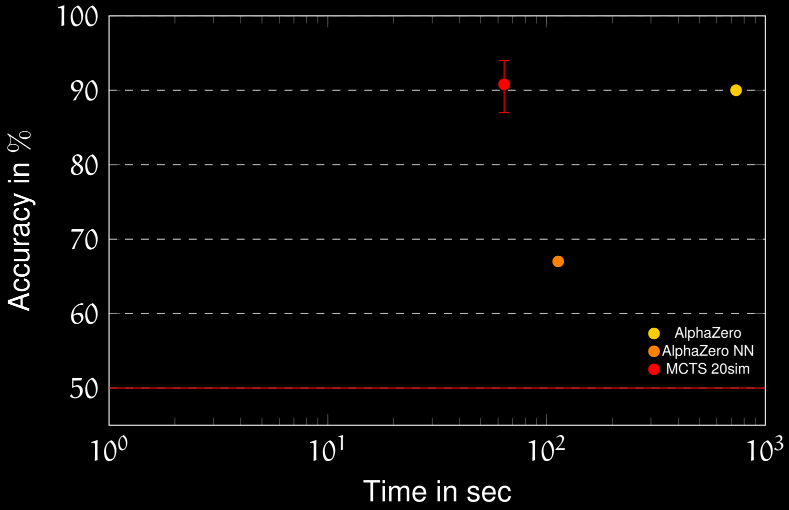

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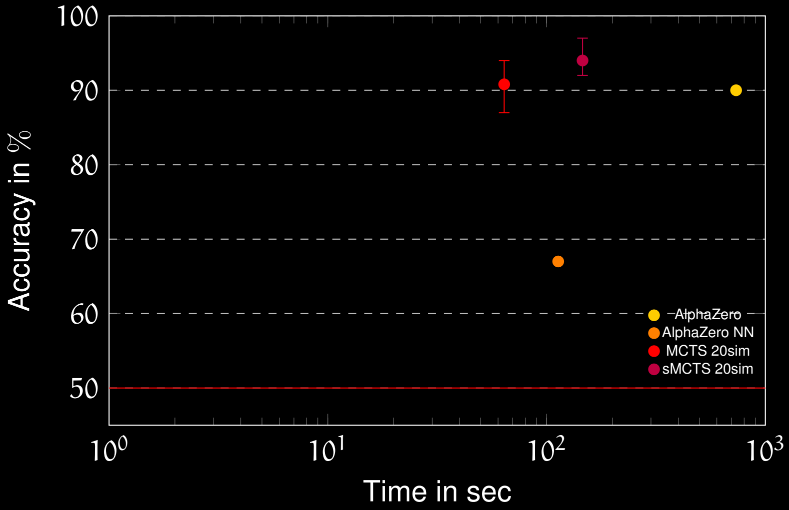

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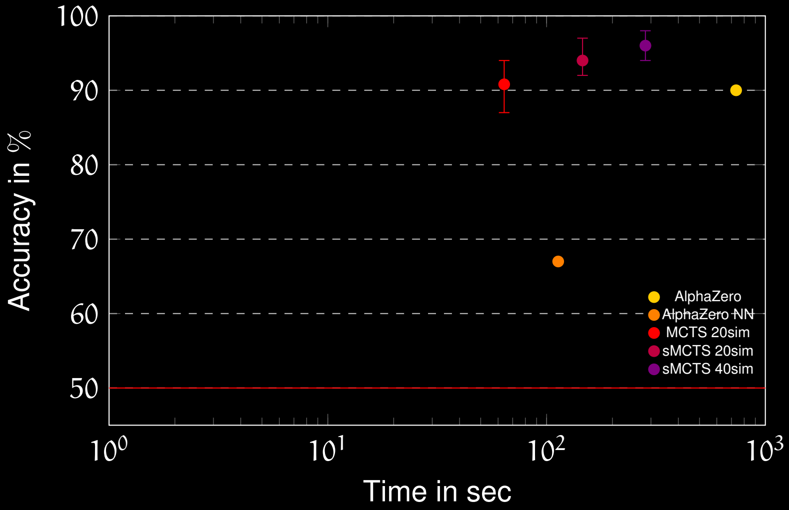

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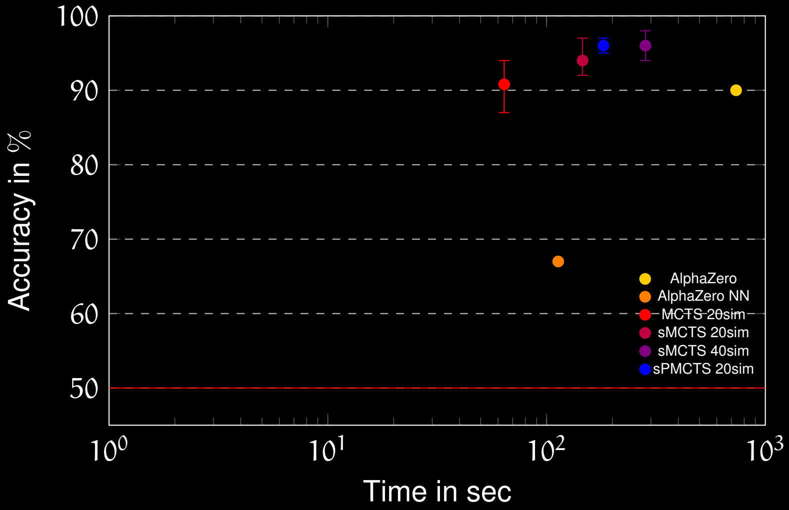

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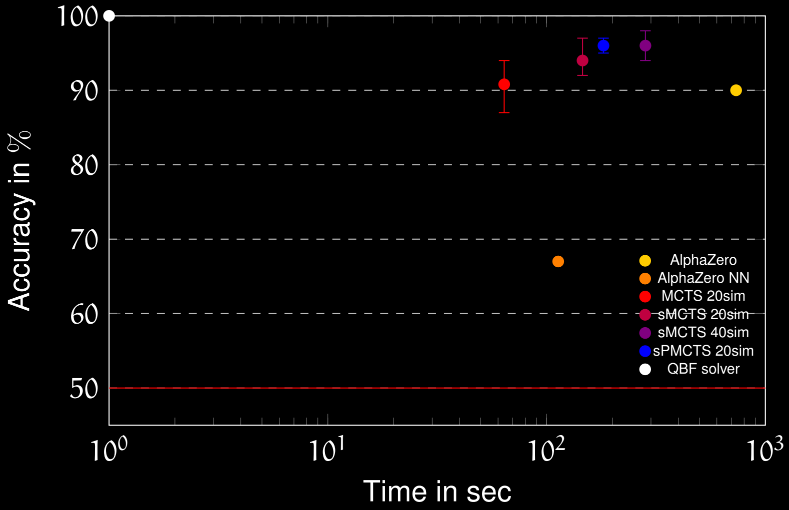

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- Try to integrate MCTS into QBF solver (e.g. to find good partial assigments)