

COMPUTING NONCOMMUTATIVE GRÖBNER BASES AND CERTIFYING OPERATOR IDENTITIES



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Institute for Algebra, JKU Linz
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Motivation

Theorem (Werner, 1994)

Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$ with inner inverses A^- and B^- . If A^-ABB^- is idempotent, then B^-A^- is an inner inverse of AB .

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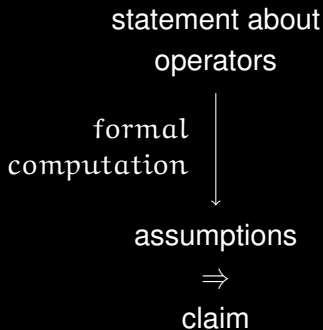
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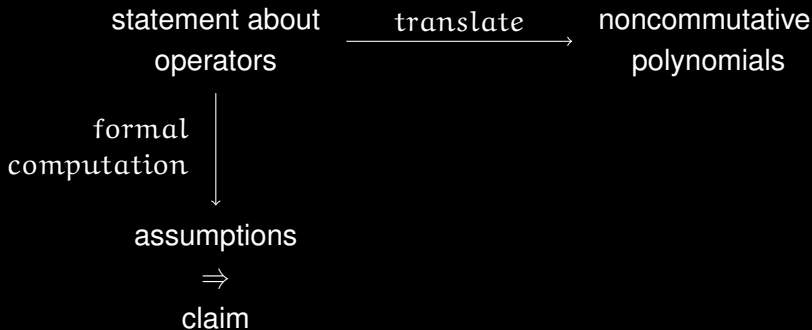


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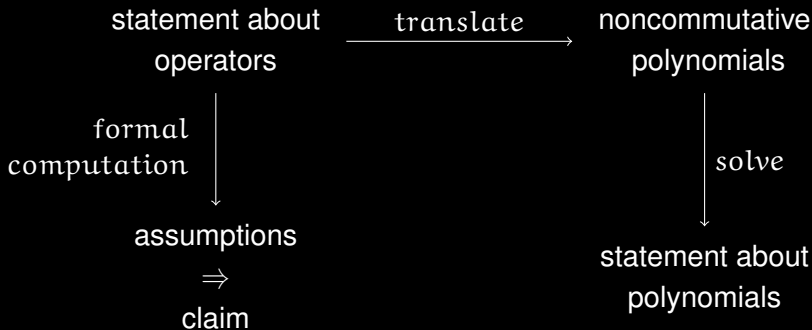


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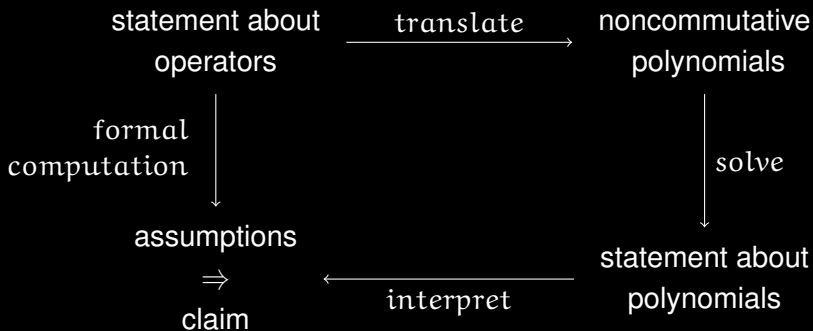


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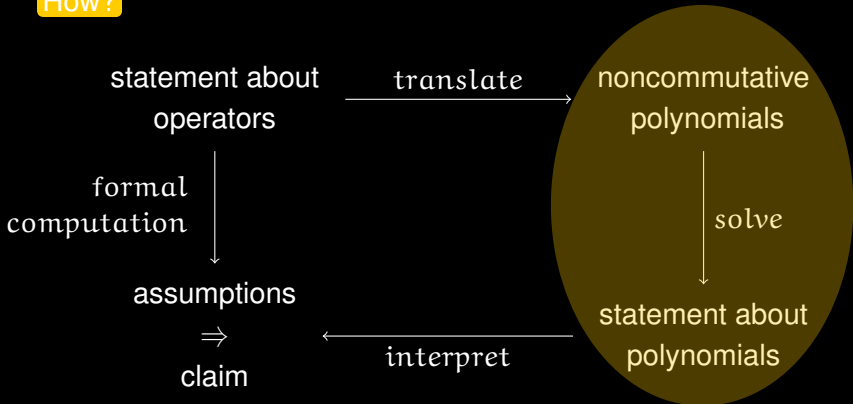


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 - Multiplication: R -bilinear extension of multiplication in $\langle X \rangle$
- For $F \subseteq R\langle X \rangle$ we denote

$$(F) = \left\{ \sum a_i f_i b_i \mid a_i, b_i \in R\langle X \rangle, f_i \in F \right\}$$

Framework for verifying operator identities

(Raab, Regensburger, Hossein Poor, 2021)

1. Phrase all assumptions on the operators involved as well as the claimed property in terms of identities.
2. Convert these identities into polynomials by uniformly replacing each operator by a unique noncommutative indeterminate in the differences of the left and right hand sides.
3. Prove that the polynomial f corresponding to the claim lies in the ideal (F) generated by the set of polynomials corresponding to the assumptions.

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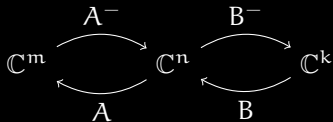
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 - 4*. Consider different settings where the assumptions still hold and immediately obtain analogous statements in those settings as well.
- translate
- solve
- interpret

More formally...

Setting is encoded in **labelled quiver** Q .

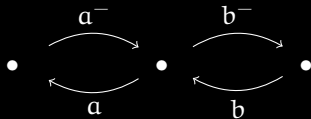
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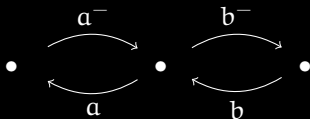
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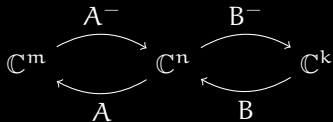


Quiver representations in an R -linear category

Vertices and edges of Q are assigned **objects and morphisms**.

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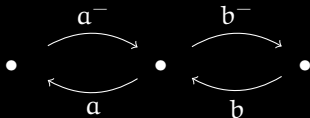


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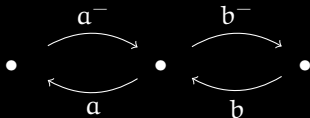
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\Rightarrow Compatible polynomials have realizations as morphisms.

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Theorem (Raab, Regensburger, Hossein Poor, 2021)

Let $F \subseteq R\langle X \rangle$ and $f \in (F)$. Then, for every labelled quiver Q and every representation of Q in an R -linear category s.t.

1. f and all elements of F are compatible with Q , and
2. realizations of all elements of F are zero,

we have that the realization of f is zero.

Example revisited

Assumptions:

$$AA^{-1} = A, \quad BB^{-1} = B, \quad (A^{-1}ABB^{-1})^2 = A^{-1}ABB^{-1}$$

Claim: $ABB^{-1}A^{-1}AB = AB$

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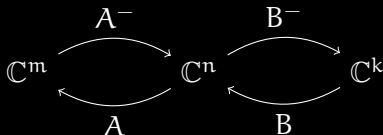
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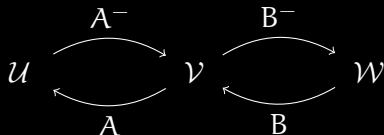
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Successfully used to **automatically (im)prove statements** in the field of

- **generalised inverses.**
- **homological algebra.**

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In the field of generalised inverses:

- Reverse order laws

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In the field of generalised inverses:

- Reverse order laws
- Solvability of systems of equations

Applications

(Milošević, 2020)

Theorem 2.2 Let a_i, b_i, c_i be elements of a ring \mathcal{R} with a unit such that a_i, b_i are regular and $a_i a_i^- c_i b_i^- b_i = c_i$ for $i = \overline{1, 3}$. Additionally, let $s = a_2 l_{a_1}, j = a_3 l_{a_1}, m = j l_s, t = r_{b_1} b_2, k = r_{b_1} b_3, n = r_t k, p = a_3 l_{a_2}, q = r_{b_2} b_3$ and $s, j, m, t, k, n, r_{mp}, ql_n, r_{r_{mp} r_m j}, ql_k l_n \in \mathcal{R}^-$. The following are equivalent:

- (i) The system of equations (11) is consistent.
(ii) The conditions

$$\begin{aligned} r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t &= 0 \\ r_{r_{mp} r_m j} (r_{r_{mp} r_m j})^- r_{r_{mp} r_m} e l_n (q l_n)^- q l_n &= r_{r_{mp} r_m} e l_n \\ r_m j j^- e l_k l_n (q l_k l_n)^- q l_k l_n &= r_m e l_k l_n \end{aligned}$$

are satisfied, where $e = c_3 - j s^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - j s^- c_2 l_t b_2^- b_3 - (a_3 - j s^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)$.

- (iii) The conditions

$$\begin{aligned} r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t &= 0 \\ r_j(c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k &= 0 \\ r_m(c_3 - j s^- c_2 l_t b_2^- b_3 - (a_3 - j s^- a_2) a_1^- c_1 b_1^- b_3) l_k l_n l_{q l_k l_n} &= 0 \\ r_{r_{mp} r_m j} r_{r_{mp} r_m} (c_3 - a_3 a_2^- r_s c_2 t^- k - a_3 a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n &= 0 \\ r_{r_{mp} r_m} (c_3 - j s^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - j s^- c_2 l_t b_2^- b_3 \\ - (a_3 - j s^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n l_{q l_n} &= 0. \end{aligned}$$

are satisfied.

In that case the general solution of (11) is given by (34), where

$$\begin{aligned}
 z_1 &= c_1 f + g^{-r_s} c_2 t^{-t} + a_1 l_{a_2} (r_m p)^{-r_m} [e - j(r_{r_m p} r_m j)^{-r_{r_m p} r_m} e \\
 &\quad - j l_{r_m p} r_m j s^{-u_3} b_1^{-1} (1 - b_2 t^{-r_{b_1}}) r_{q l_k l_n} q - (1 - j(r_{r_m p} r_m j)^{-r_{r_m p} r_m}) e l_k l_n (q l_k l_n)^{-q} l_n k^{-t} \\
 &\quad + a_1 a_1^{-1} l_g u_1 t t^{-t} - a_1 l_{a_2} (r_m p)^{-r_m} p (1 - l_{a_1} s^{-a_2}) a_1^{-1} u_1 t^{-k} l_n k^{-t}, \\
 z_2 &= (c_1 b_1^{-1} (b_3 - b_2 t^{-k}) + g^{-r_s} c_2 t^{-k}) l_n + a_1 a_3^{-1} c_3 n^{-n} + a_1 (1 - a_3^{-1} r_m r_j a_3) a_1^{-1} u_2 n^{-n} \\
 &\quad a_1 l_{a_2} (r_m p)^{-r_m} [e - j(r_{r_m p} r_m j)^{-r_{r_m p} r_m} e - j l_{r_m p} r_m j s^{-u_3} b_1^{-1} (1 - b_2 t^{-r_{b_1}}) r_{q l_k l_n} q \\
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 &\quad - a_1 l_{a_2} (r_m p)^{-r_m} p (1 - l_{a_1} s^{-a_2}) a_1^{-1} u_1 t^{-k} l_n, \\
 z_3 &= g c_1 + s s^{-c_2} l_t f^{-t} + s (r_{r_m p} r_m j)^{-r_{r_m p} r_m} e l_n (q l_n)^{-r_{b_2} b_1} \\
 &\quad + s (l_{r_m p} r_m j j^{-t} + (1 - j^{-r_m j}) (r_{r_m p} r_m j)^{-r_{r_m p} r_m}) r_m e l_k l_n (q l_k l_n)^{-r_{b_2} b_1} \\
 &\quad + s s^{-u_3} r_f b_1^{-1} b_1 - s j^{-r_m j} l_{r_m p} r_m j s^{-u_3} b_1^{-1} (1 - b_2 t^{-r_{b_1}}) q l_k l_n (q l_k l_n)^{-r_{b_2} b_1} \\
 &\quad - s (r_{r_m p} r_m j)^{-r_{r_m p} r_m} j s^{-u_3} b_1^{-1} (1 - b_2 t^{-r_{b_1}}) q l_n (q l_n)^{-r_{b_2} b_1}, \\
 z_4 &= g c_1 b_1^{-1} (b_3 - b_2 t^{-k}) l_n + s s^{-c_2} l_t b_2^{-1} b_3 l_n + (g g^{-r_s} c_2 + s s^{-c_2}) t^{-k} l_n + r_s a_2 a_3^{-1} c_3 n^{-n} \\
 &\quad + r_s a_2 (1 - a_3^{-1} r_m r_j a_3) a_1^{-1} u_2 n^{-n} + s j^{-r_m j} [s^{-a_2} a_3^{-1} c_3 + (s^{-a_2} (1 - a_3^{-1} r_m r_j a_3) - j^{-a_3}) a_1^{-1} u_2] n^{-n} \\
 &\quad + s (1 - j^{-r_m j}) s^{-u_4} n^{-n} + s (r_{r_m p} r_m j)^{-r_{r_m p} r_m} e l_n (q l_n)^{-r_{b_2} b_1} b_1^{-1} (b_3 - b_2 t^{-k}) l_n \\
 &\quad + s (l_{r_m p} r_m j j^{-t} + (1 - j^{-r_m j}) (r_{r_m p} r_m j)^{-r_{r_m p} r_m}) r_m e l_k l_n (q l_k l_n)^{-r_{b_2} b_1} b_1^{-1} (b_3 - b_2 t^{-k}) l_n \\
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 z_5 &= r_m ((a_3 - j s^{-a_2}) a_1^{-1} c_1 + j s^{-c_2} l_t f^{-t}) + m m^{-c_3} b_3^{-1} b_1 + m m^{-u_5} b_1^{-1} (1 - b_3 l_k l_n b_3^{-1}) b_1 \\
 &\quad + r_m j (r_{r_m p} r_m j)^{-r_{r_m p} r_m} e l_n (q l_n)^{-r_{b_2} b_1} + r_m j l_{r_m p} r_m j^{-r_m} e l_k l_n (q l_k l_n)^{-r_{b_2} b_1} + r_m j s^{-u_3} r_f b_1^{-1} b_1 \\
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 z_6 &= r_m (a_3 - j s^{-a_2}) a_1^{-1} c_1 f + r_m a_3 a_2^{-1} r_s c_2 t^{-t} + r_m j s^{-c_2} l_t f^{-t} + c_2 t^{-t}) + m m^{-c_3} b_3^{-1} b_2 l_t \\
 &\quad + m m^{-u_5} b_1^{-1} (1 - b_3 l_k l_n b_3^{-1}) b_2 l_t + m m^{-c_3} b_3^{-1} b_2 t^{-t} + u_5 b_1^{-1} ((1 - b_3 l_k l_n b_3^{-1}) b_2 t^{-t} - b_3 k^{-k}) l_n k^{-t} \\
 &\quad + m m^{-u_6} t^{-t} (1 - k l_n k^{-t}) + r_m (a_3 - j s^{-a_2}) a_1^{-1} l_g u_1 t^{-t} \\
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 &\quad - j (r_{r_m p} r_m j)^{-r_{r_m p} r_m} e - (1 - j(r_{r_m p} r_m j)^{-r_{r_m p} r_m}) e l_k l_n (q l_k l_n)^{-q} l_n k^{-t} \\
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 \end{aligned}$$

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- Solvability of systems of equations

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Any further ideas for possible applications?

NONCOMMUTATIVE GRÖBNER BASES



Polynomial reduction

We fix a monomial ordering \preceq on $\langle X \rangle$.

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Definition

Let $f, f', g \in K\langle X \rangle$ with $g \neq 0$. We say that f **reduces** to f' by g if there exist $a, b \in \langle X \rangle$ such that $a \operatorname{lm}(g)b \in \operatorname{supp}(f)$ and

$$f' = f - \frac{\operatorname{coeff}(f, a \operatorname{lm}(g)b)}{\operatorname{lc}(g)} \cdot agb.$$

In this case, we write $f \rightarrow_g f'$,

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Adapt to sets $G \subseteq K\langle X \rangle$ by

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We denote by \rightarrow_G^* the **reflexive, transitive closure** of \rightarrow_G .

Gröbner bases

Definition

Let $I \subseteq K\langle X \rangle$ be an ideal and $G \subseteq I$ such that $(G) = I$. Then, G is called a **Gröbner basis** of I if and only if \rightarrow_G is confluent.

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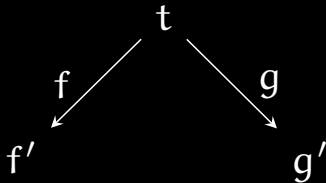
- every $f \in K\langle X \rangle$ has a **unique normal form** under \rightarrow_G .
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Caution

Not all ideals in $K\langle X \rangle$ have a finite Gröbner basis!

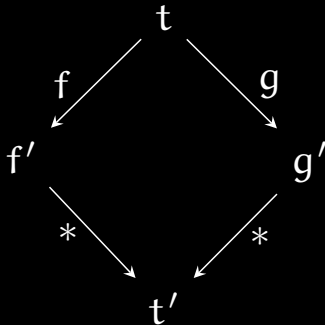
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If we want to check whether $G \subseteq K\langle X \rangle$ is a Gröbner basis, we have to consider these branching points for all $f, g \in G$:



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Ambiguities

Definition

Let $f, g \in G \subseteq K\langle X \rangle$ be such that $f, g \neq 0$.

If $\text{lm}(f) = AB$ and $\text{lm}(g) = BC$ for some $A, B, C \in \langle X \rangle \setminus \{1\}$.

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If $\text{lm}(f) = ABC$ and $\text{lm}(g) = B$ for some $A, B, C \in \langle X \rangle$. Then, we call (ABC, A, C, f, g) an **inclusion ambiguity** of G . Its **critical pair** is $(ABC - f', ABC - g')$ with $ABC \xrightarrow{1, f, 1} f'$ and $ABC \xrightarrow{A, g, C} g'$.

Diamond Lemma

Definition

Let $G \subseteq K\langle X \rangle$ and let $\alpha = (ABC, A, C, f, g)$ be an ambiguity of $f, g \in G \setminus \{0\}$. Furthermore, let $(ABC - f', ABC - g')$ be the critical pair of α . Then, the **S-polynomial** of α is given by

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Diamond Lemma (Bergman, 1978)

Let $G \subseteq K\langle X \rangle$. Then, G is a Gröbner basis of (G) if and only if $\text{spol}(\alpha) \rightarrow_G^* 0$ for all ambiguities α of G .

Buchberger algorithm

Input: a finite set $F \subseteq K\langle X \rangle$

Output if the algorithm terminates: $G \subseteq K\langle X \rangle$ such that G is a Gröbner basis of (F)

- 1: $G \leftarrow F$
 - 2: $\text{amb} \leftarrow$ all ambiguities of G
 - 3: **while** $\text{amb} \neq \emptyset$ **do**
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 - 5: select $a \in \text{amb}$
 - 6: $\text{amb} \leftarrow \text{amb} \setminus \{a\}$
 - 7: compute a normal form f' of $\text{spol}(a)$ w.r.t. to \rightarrow_G
 - 8: **if** $f' \neq 0$ **then**
 - 9: $G \leftarrow G \cup \{f'\}$
 - 10: $\text{amb} \leftarrow$ all new ambiguities of G
 - 11: **return** G
-

(NON)COMMUTATIVE F4



Faugère's F4 algorithm

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A new efficient algorithm for computing Gröbner bases (F4).
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- Noncommutative F4 first published by Xingqiang Xiu in *Non-Commutative Gröbner Bases and Applications*.
PhD thesis, University of Passau, 2012.

Polynomials \longleftrightarrow matrices

Let $F = \{f_1, f_2, f_3\} \subseteq \mathbb{Q}\langle x, y, z \rangle$, with

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$$\underbrace{\begin{array}{cccc} xxyz & xyy & yz & x \\ \left(\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \end{array} \right) & \begin{array}{l} f_1 \\ f_2 \\ f_3 \end{array} & \longrightarrow & \left(\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array} \right) \end{array}}_{= \mathcal{M}_F}$$

Polynomials \longleftrightarrow matrices

Let $F = \{f_1, f_2, f_3\} \subseteq \mathbb{Q}\langle x, y, z \rangle$, with

$$f_1 = xxyz + 2xyy + x, \quad f_2 = xyy - yz, \quad f_3 = yz - 2x.$$

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$$\mathcal{F}_{\text{RRef}(\mathcal{M}_F)} = \{xxyz + 5x, xyy - 2x, yz - 2x\}$$

First idea

Given $G = \{g_1, \dots, g_k\} \subseteq K\langle X \rangle$ and $F = \{f_1, \dots, f_m\} \subseteq K\langle X \rangle$ we want to reduce all $f \in F$ by G simultaneously.

First idea

Given $G = \{g_1, \dots, g_k\} \subseteq K\langle X \rangle$ and $F = \{f_1, \dots, f_m\} \subseteq K\langle X \rangle$ we want to reduce all $f \in F$ by G simultaneously.

$$\mathcal{M}_{\text{GUF}} = \left(\begin{array}{cccc|c} * & \dots & \dots & * & g_1 \\ \vdots & & & \vdots & \vdots \\ * & \dots & \dots & * & g_k \\ \hline * & \dots & \dots & * & f_1 \\ \vdots & & & \vdots & \vdots \\ * & \dots & \dots & * & f_m \end{array} \right)$$

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Symbolic preprocessing

Input: a finite set $F \subseteq K\langle X \rangle$ and $G \subseteq K\langle X \rangle$

Output: $G' \subseteq \{agb \mid a, b \in \langle X \rangle, g \in G\}$

- 1: $G' \leftarrow \emptyset$
 - 2: $T \leftarrow \text{supp}(F) \setminus \text{lm}(F)$
 - 3: $\text{done} \leftarrow \text{lm}(F)$
 - 4: **while** $T \neq \emptyset$ **do**
 - 5: **select** $t \in T$
 - 6: $T \leftarrow T \setminus \{t\}$
 - 7: $\text{done} \leftarrow \text{done} \cup \{t\}$
 - 8: **if** there exist $g \in G, a, b \in \langle X \rangle$ s.t. $t = a \text{lm}(g)b$ **then**
 - 9: $G' \leftarrow G' \cup \{agb\}$
 - 10: $T \leftarrow T \cup (\text{supp}(agb) \setminus \text{done})$
 - 11: **return** G'
-

Symbolic preprocessing

Input: a finite set $F \subseteq K[X]$ and $G \subseteq K[X]$

Output: $G' \subseteq \{tg \mid t \in [X], g \in G\}$

- 1: $G' \leftarrow \emptyset$
 - 2: $T \leftarrow \text{supp}(F) \setminus \text{lm}(F)$
 - 3: $\text{done} \leftarrow \text{lm}(F)$
 - 4: **while** $T \neq \emptyset$ **do**
 - 5: **select** $t \in T$
 - 6: $T \leftarrow T \setminus \{t\}$
 - 7: $\text{done} \leftarrow \text{done} \cup \{t\}$
 - 8: **if** there exist $g \in G$, $t' \in [X]$ s.t. $t = t' \text{lm}(g)$ **then**
 - 9: $G' \leftarrow G' \cup \{t'g\}$
 - 10: $T \leftarrow T \cup (\text{supp}(t'g) \setminus \text{done})$
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Symbolic preprocessing

Let $(ABC - f', ABC - g')$ be the critical pair of an ambiguity of G .

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$$\mathcal{M}_{F'} = \begin{pmatrix} 1 & * & \cdots & * \\ 1 & * & \cdots & * \\ \hline & * & \cdots & * \\ & \vdots & & \vdots \\ & * & \cdots & * \end{pmatrix} \begin{array}{l} ABC - f' \\ ABC - g' \\ g_1 \\ \vdots \\ g_m \end{array}$$

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Example

Let $F = \{f_1, f_2, f_3, f_4\} \subseteq \mathbb{Q}\langle x, y \rangle$, with

$$f_1 = yxyx,$$

$$f_2 = yxyx + yxx,$$

$$f_3 = yxy,$$

$$f_4 = yxy + xy.$$

Let $G = \{g_1, g_2, g_3\} \subseteq \mathbb{Q}\langle x, y \rangle$, with

$$g_1 = yx + x, \quad g_2 = y + 1, \quad g_3 = yxy$$

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Reduction

Input: a finite set $F \subseteq K\langle X \rangle$ and $G \subseteq K\langle X \rangle$

Output: $\tilde{F} \subseteq K\langle X \rangle \setminus \{0\}$

- 1: $G' \leftarrow \text{SymbolicPreprocessing}(F, G)$
 - 2: $F' \leftarrow F \cup G'$
 - 3: $\tilde{F} \leftarrow \{f \in \mathcal{F}_{\text{RRef}(\mathcal{M}_{F'})} \mid f \neq 0 \text{ and } \text{lm}(f) \notin \text{lm}(F')\}$
 - 4: **return** \tilde{F}
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Reduction

Input: a finite set $F \subseteq \mathbf{K}[X]$ and $G \subseteq \mathbf{K}[X]$

Output: $\tilde{F} \subseteq \mathbf{K}[X] \setminus \{0\}$

- 1: $G' \leftarrow \text{SymbolicPreprocessing}(F, G)$
 - 2: $F' \leftarrow F \cup G'$
 - 3: $\tilde{F} \leftarrow \{f \in \mathcal{F}_{\text{RRef}(\mathcal{M}_{F'})} \mid f \neq 0 \text{ and } \text{lm}(f) \notin \text{lm}(F')\}$
 - 4: **return** \tilde{F}
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Main result

Theorem

Let $G \subseteq K\langle X \rangle$ and let C be a finite subset of critical pairs of ambiguities of G . Furthermore, let $F = \bigcup_{(\tilde{f}, \tilde{g}) \in C} \{\tilde{f}, \tilde{g}\}$ be the set of all polynomials appearing in the critical pairs of C and let $\tilde{F} = \text{Reduction}(F, G)$. Then,

$$\text{spol}(\alpha) \xrightarrow{*}_{G\tilde{F}} 0,$$

for all ambiguities α of G such that the critical pair of α is in C .

F4 algorithm

Input: a finite set $\{f_1, \dots, f_m\} \subseteq K\langle X \rangle$

Output if algorithm terminates: $G \subseteq K\langle X \rangle$ GB of (f_1, \dots, f_m)

- 1: $G \leftarrow \{f_1, \dots, f_m\} \setminus \{0\}$
 - 2: $\text{critPairs} \leftarrow$ critical pairs of all ambiguities of G
 - 3: **while** $\text{critPairs} \neq \emptyset$ **do**
 - 4: **while** $\text{critPairs} \neq \emptyset$ **do**
 - 5: select $C \subseteq \text{critPairs}$
 - 6: $\text{critPairs} \leftarrow \text{critPairs} \setminus C$
 - 7: $F \leftarrow \bigcup_{(\tilde{f}, \tilde{g}) \in C} \{\tilde{f}, \tilde{g}\}$
 - 8: $\tilde{F} \leftarrow \text{Reduction}(F, G)$
 - 9: $G \leftarrow G \cup \tilde{F}$
 - 10: $\text{critPairs} \leftarrow$ critical pairs of all new ambiguities of G
 - 11: **return** G
-

F4 algorithm

Input: a finite set $\{f_1, \dots, f_m\} \subseteq K[X]$

Output: $G \subseteq K[X]$ GB of (f_1, \dots, f_m)

- 1: $G \leftarrow \{f_1, \dots, f_m\} \setminus \{0\}$
 - 2: $\text{critPairs} \leftarrow$ all critical pairs of G
 - 3: **while** $\text{critPairs} \neq \emptyset$ **do**
 - 4: **while** $\text{critPairs} \neq \emptyset$ **do**
 - 5: select $C \subseteq \text{critPairs}$
 - 6: $\text{critPairs} \leftarrow \text{critPairs} \setminus C$
 - 7: $F \leftarrow \bigcup_{(\tilde{f}, \tilde{g}) \in C} \{\tilde{f}, \tilde{g}\}$
 - 8: $\tilde{F} \leftarrow \text{Reduction}(F, G)$
 - 9: $G \leftarrow G \cup \tilde{F}$
 - 10: $\text{critPairs} \leftarrow$ all new critical pairs of G
 - 11: **return** G
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Example

Let

$$f_1 = yx + x, \quad f_2 = y + 1, \quad f_3 = yxy \quad \in \mathbb{Q}\langle x, y \rangle.$$

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$$G = \{f_1, f_2, f_3\}$$

$$a_{12} = (yx, 1, x, f_1, f_2),$$

$$a_{31} = (yxy, 1, y, f_3, f_1),$$

$$a'_{31} = (yxyx, yx, x, f_3, f_1),$$

$$a_{32} = (yxy, yx, 1, f_3, f_2),$$

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$$\begin{aligned} a_{12} &= (yx, 1, x, f_1, f_2), & a_{31} &= (yxy, 1, y, f_3, f_1), \\ a'_{31} &= (yxyx, yx, x, f_3, f_1), & a_{32} &= (yxy, yx, 1, f_3, f_2), \\ a'_{32} &= (yxy, 1, xy, f_3, f_2), & a_{33} &= (yxyxy, yx, xy, f_3, f_3) \end{aligned}$$

$$F = \{yxyx, yxyx + yxx, yxy, yxy + xy\}$$

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Let

$$f_1 = yx + x, \quad f_2 = y + 1, \quad f_3 = yxy \quad \in \mathbb{Q}\langle x, y \rangle.$$

$$G = \{f_1, f_2, f_3\}$$

$$\begin{aligned} a_{12} &= (yx, 1, x, f_1, f_2), & a_{31} &= (yxy, 1, y, f_3, f_1), \\ a'_{31} &= (yxyx, yx, x, f_3, f_1), & a_{32} &= (yxy, yx, 1, f_3, f_2), \\ a'_{32} &= (yxy, 1, xy, f_3, f_2), & a_{33} &= (yxyxy, yx, xy, f_3, f_3) \end{aligned}$$

$$F = \{yxyx, yxyx + yxx, yxy, yxy + xy\}$$

$$\xrightarrow{\text{SymPre}} F' = F \cup \{f_1x, xf_2\}$$

Example

$$\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \\ \underbrace{\hspace{10em}} \\ = \mathcal{M}_F' \end{array}$$

Example

$$\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}}_{= \mathcal{M}_F'}$$

Example

$$\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$= \mathcal{M}_{F'}$

Example

$$\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$= \mathcal{M}_{F'}$

Example

$$\begin{array}{cccccc}
 yxyx & yxy & yxx & xy & xx & x \\
 \left(\begin{array}{cccccc}
 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1
 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccccc}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right)
 \end{array}$$

$\underbrace{\hspace{15em}}_{= \mathcal{M}_{\tilde{F}'}}$

$$\implies \tilde{F} = \{xx, x\}$$

Example

$$\begin{array}{cccccc}
 yxyx & yxy & yxx & xy & xx & x \\
 \left(\begin{array}{cccccc}
 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1
 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccccc}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right)
 \end{array}$$

$\underbrace{\hspace{15em}}_{= \mathcal{M}_{\tilde{F}'}}$

$$\implies \tilde{F} = \{xx, x\}$$

$$G = \{f_1, f_2, f_3\} \cup \{xx, x\}$$

Example

$$f_1 = yx + x,$$

$$f_2 = y + 1,$$

$$f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

Example

$$f_1 = yx + x,$$

$$f_2 = y + 1,$$

$$f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

$$a_{14} = (yx, y, x, f_1, xx),$$

$$a_{15} = (yx, y, 1, f_1, x),$$

$$a_{35} = (yxy, y, y, f_3, x),$$

$$a_{44} = (xxx, x, x, xx, xx),$$

$$a_{45} = (xx, x, 1, xx, x),$$

$$a'_{45} = (xx, 1, x, x, xx).$$

Example

$$f_1 = yx + x, \quad f_2 = y + 1, \quad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

$$a_{14} = (yxx, y, x, f_1, xx),$$

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$$a_{44} = (xxx, x, x, xx, xx),$$

$$a_{45} = (xx, x, 1, xx, x),$$

$$a'_{45} = (xx, 1, x, x, xx).$$

$$F = \{yx + xx, yxx, yx + x, yx\}$$

Example

$$f_1 = yx + x, \quad f_2 = y + 1, \quad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

$$a_{14} = (yx, y, x, f_1, xx),$$

$$a_{15} = (yx, y, 1, f_1, x),$$

$$a_{35} = (yxy, y, y, f_3, x),$$

$$a_{44} = (xxx, x, x, xx, xx),$$

$$a_{45} = (xx, x, 1, xx, x),$$

$$a'_{45} = (xx, 1, x, x, xx).$$

$$F = \{yx + xx, yxx, yx + x, yx\}$$

$$\xrightarrow{\text{SymPre}} F' = F \cup \{xx, x\}$$

Example

$$\begin{array}{cccc} & yxx & yx & xx & x \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \\ \underbrace{\hspace{10em}} & & & & \\ & = \mathcal{M}_{F'} & & & \end{array}$$

Example

$$\begin{array}{cccc} & yxx & yx & xx & x \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

$\underbrace{\hspace{15em}}_{= \mathcal{M}_{F'}}$

Example

$$\underbrace{\begin{array}{c} \text{yxx} \quad \text{yx} \quad \text{xx} \quad \text{x} \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \end{array}}_{= \mathcal{M}_{F'}} \xrightarrow{\text{RRef}} \begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

Example

$$\underbrace{\begin{array}{c} \text{yx} \quad \text{yx} \quad \text{xx} \quad \text{x} \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \end{array}}_{= \mathcal{M}_{F'}} \xrightarrow{\text{RRef}} \begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

Example

$$\begin{array}{cccc} & yxx & yx & xx & x \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

$\underbrace{\hspace{15em}}_{= \mathcal{M}_{F'}} \implies \tilde{F} = \emptyset$

Example

$$\begin{array}{cccc} & yxx & yx & xx & x \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) & \xrightarrow{\text{RRef}} & \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ \underbrace{\hspace{10em}} & = \mathcal{M}_{F'} & & & \\ & \implies \tilde{F} = \emptyset & & & \end{array}$$

$G = \{f_1, f_2, f_3, xx, x\}$ is a GB of (f_1, f_2, f_3)

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Framework to algebraically prove operator identities:

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