COMPUTING NONCOMMUTATIVE GRÖBNER BASES AND CERTIFYING OPERATOR IDENTITIES

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Der Wissenschaftsfonds.

Motivation

Theorem (Werner, 1994)

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$ with inner inverses A^- and B^-. If A^-ABB^- is idempotent, then B^-A^- is an inner inverse of AB.

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  - Addition: term wise
  - Multiplication: R-bilinear extension of multiplication in $\langle X \rangle$
- For $F \subseteq R\langle X \rangle$ we denote

$$(F) = \{\sum a_i f_i b_i \mid a_i, b_i \in R\langle X \rangle,\ f_i \in F\}$$

Framework for verifying operator identities

(Raab, Regensburger, Hossein Poor, 2021)

1. Phrase all assumptions on the operators involved as well as the claimed property in terms of identities.
2. Convert these identities into polynomials by uniformly replacing each operator by a unique noncommutative indeterminate in the differences of the left and right hand sides.
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4*. Consider different settings where the assumptions still hold and immediately obtain analogous statements in those settings as well.

translate

solve

interpret

More formally. . .

Setting is encoded in labelled quiver Q.

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Vertices and edges of Q are assigned objects and morphisms.

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$f \in R\langle X\rangle$ compatible with $Q \iff$ monomials of f are paths in Q with same start and end.

$\Rightarrow$ Compatible polynomials have realizations as morphisms.

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Theorem (Raab, Regensburger, Hossein Poor, 2021)

Let $F \subseteq R\langle X \rangle$ and $f \in (F)$. Then, for every labelled quiver Q and every representation of Q in an R-linear category s.t.

1. f and all elements of F are compatible with Q, and
2. realizations of all elements of F are zero,

we have that the realization of f is zero.

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In $\mathbb{Q}\langle a, b, a^{-}, b^{-}\rangle$:

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Then, "assumptions $\Rightarrow$ claim" since

$$f = f_1(bb^{-}b - bb^{-}a^{-}abb^{-}b) + (a - abb^{-}a^{-}a)f_2 + af_3b.$$

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Framework implemented in the MATHEMATICA and SAGEMATH package `OperatorGB`. Available at

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Successfully used to automatically (im)prove statements in the field of

- generalised inverses.
- homological algebra.

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- Reverse order laws

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In the field of generalised inverses:

- Reverse order laws
- Solvability of systems of equations

Applications

(Milošević, 2020)

Theorem 2.2 *Let a_i, b_i, c_i be elements of a ring $\mathcal{R}$ with a unit such that a_i, b_i are regular and $a_i a_i^- c_i b_i^- b_i = c_i$ for $i = \overline{1,3}$. Additionally, let $s = a_2 l_{a_1}, j = a_3 l_{a_1}, m = j l_s, t = r_{b_1} b_2, k = r_{b_1} b_3, n = r_t k, p = a_3 l_{a_2}, q = r_{b_2} b_3$ and $s, j, m, t, k, n, r_m p, q l_n, r_{r_m p} r_m j, q l_k l_n \in \mathcal{R}^-$. The following are equivalent:*

(*i*) *The system of equations* (11) *is consistent.*

(*ii*) *The conditions*

$$
\begin{aligned}
&r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0 \\
&r_{r_m p} r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- q l_n = r_{r_m p} r_m e l_n \\
&r_m j j^- e l_k l_n (q l_k l_n)^- q l_k l_n = r_m e l_k l_n
\end{aligned}
$$

are satisfied, where $e = c_3 - js^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)$.

(*iii*) *The conditions*

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\begin{aligned}
&r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0 \\
&r_j(c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k = 0 \\
&r_m(c_3 - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2) a_1^- c_1 b_1^- b_3) l_k l_n l_{q l_k l_n} = 0 \\
&r_{r_{r_m p} r_m j} r_{r_m p} r_m (c_3 - a_3 a_2^- r_s c_2 t^- k - a_3 a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n = 0 \\
&r_{r_m p} r_m (c_3 - js^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - js^- c_2 l_t b_2^- b_3 \\
&-(a_3 - js^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n l_{q l_n} = 0.
\end{aligned}
$$

are satisfied.

In that case the general solution of (11) *is given by* (34), *where*

$$
\begin{aligned}
z_1 = {} & c_1 f + g^- r_s c_2 t^- t + a_1 l_{a_2} (r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e \\
& - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t \\
& + a_1 a_1^- l_g u_1 t t^- - a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t, \\
z_2 = {} & (c_1 b_1^- (b_3 - b_2 t^- k) + g^- r_s c_2 t^- k) l_n + a_1 a_3^- c_3 n^- n + a_1 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n \\
& a_1 l_{a_2} (r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
& - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] k^- k l_n + a_1 a_1^- l_g u_1 t^- k l_n \\
& - a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n, \\
z_3 = {} & g c_1 + s s^- c_2 l_t f^- + s (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 \\
& + s (l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
& + s s^- u_3 r_f b_1^- b_1 - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
& - s (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1, \\
z_4 = {} & g c_1 b_1^- (b_3 - b_2 t^- k) l_n + + s s^- c_2 l_t b_2^- b_3 l_n + (g g^- r_s c_2 + s s^- c_2) t^- k l_n + r_s a_2 a_3^- c_3 n^- n \\
& + r_s a_2 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n + s j^- r_m j [s^- a_2 a_3^- c_3 + (s^- a_2 (1 - a_3^- r_m r_j a_3) - j^- a_3) a_1^- u_2] n^- n \\
& + s (1 - j^- r_m j) s^- u_4 n^- n + s (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& + s (l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& + s s^- u_3 r_f b_1^- (b_3 - b_2 t^- k) l_n \\
& - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n \\
& - s (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1 b_1^- (b_3 - b_2 t^- k) l_n, \\
z_5 = {} & r_m ((a_3 - j s^- a_2) a_1^- c_1 + j s^- c_2 l_t f^-) + m m^- c_3 b_3^- b_1 + m m^- u_5 b_1^- (1 - b_3 l_k l_n b_3^-) b_1 \\
& + r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 + r_m j l_{r_{r_m p} r_m j} j^- r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 + r_m j s^- u_3 r_f b_1^- b_1 \\
& - r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
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z_6 = {} & r_m (a_3 - j s^- a_2) a_1^- c_1 f + r_m a_3 a_2^- r_s c_2 t^- t + r_m j s^- (c_2 l_t f^- f + c_2 t^- t) + m m^- c_3 b_3^- b_2 l_t \\
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& + m m^- u_6 t^- (1 - k l_n k^-) t + r_m (a_3 - j s^- a_2) a_1^- l_g u_1 t^- t \\
& + r_m (a_3 - j s^- a_2) a_1^- a_1 l_{a_2} (r_m p)^- r_m [e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
& - j (r_{r_m p} r_m j)^- r_{r_m p} r_m e - (1 - j (r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t \\
& - r_m (a_3 - j s^- a_2) a_1^- a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t,
\end{aligned}
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In the field of generalised inverses:

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- Solvability of systems of equations

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Any further ideas for possible applications?

NONCOMMUTATIVE GRÖBNER BASES

Polynomial reduction

We fix a monomial ordering $\preceq$ on $\langle X \rangle$.

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$$f' = f - \frac{\mathrm{coeff}(f, a\,\mathrm{lm}(g)b)}{\mathrm{lc}(g)} \cdot agb.$$

In this case, we write $f \to_g f'$,

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We denote by $\overset{*}{\to}_G$ the reflexive, transitive closure of $\to_G$.

Gröbner bases

Definition

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Caution

Not all ideals in $K\langle X\rangle$ have a finite Gröbner basis!

Confluence check

If we want to check whether $G \subseteq K\langle X\rangle$ is a Gröbner basis, we have to consider these branching points for all $f, g \in G$:

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Let $f, g \in G \subseteq K\langle X\rangle$ be such that $f, g \neq 0$.

If $\mathrm{lm}(f) = AB$ and $\mathrm{lm}(g) = BC$ for some $A, B, C \in \langle X\rangle \setminus \{1\}$. Then, we call (ABC, A, C, f, g) an overlap ambiguity of G.

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Diamond Lemma

Definition

Let $G \subseteq K\langle X\rangle$ and let $a = (ABC, A, C, f, g)$ be an ambiguity of $f, g \in G \setminus \{0\}$. Furthermore, let $(ABC - f', ABC - g')$ be the critical pair of a. Then, the S-polynomial of a is given by

$$\mathrm{spol}(a) = (ABC - g') - (ABC - f') = f' - g'.$$

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Diamond Lemma (Bergman, 1978)

Let $G \subseteq K\langle X \rangle$. Then, G is a Gröbner basis of (G) if and only if $\mathrm{spol}(a) \rightarrow_G^* 0$ for all ambiguities a of G.

Buchberger algorithm

Input: a finite set $F \subseteq K\langle X\rangle$

Output if the algorithm terminates: $G \subseteq K\langle X\rangle$ such that G is a Gröbner basis of (F)

$G \leftarrow F$
$\mathrm{amb} \leftarrow$ all ambiguities of G
while $\mathrm{amb} \neq \emptyset$ **do**
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    select $a \in \mathrm{amb}$
    $\mathrm{amb} \leftarrow \mathrm{amb} \setminus \{a\}$
    compute a normal form f' of $\mathrm{spol}(a)$ w.r.t. to $\to_G$
    if $f' \neq 0$ **then**
      $G \leftarrow G \cup \{f'\}$
  $\mathrm{amb} \leftarrow$ all new ambiguities of G
return G

(NON)COMMUTATIVE F4

Faugère's F4 algorithm

- First published in 1999 by Jean-Charles Faugère in

  A new efficient algorithm for computing Gröbner bases (F4).
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- Noncommutative F4 first published by Xingqiang Xiu in

  Non-Commutative Gröbner Bases and Applications.
  PhD thesis, University of Passau, 2012.

Polynomials $\longleftrightarrow$ matrices

Let $F = \{f_1, f_2, f_3\} \subseteq \mathbb{Q}\langle x, y, z\rangle$, with

$$f_1 = xxyz + 2xyy + x, \qquad f_2 = xyy - yz, \qquad f_3 = yz - 2x.$$

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$$xxyz \succ_{\mathrm{dlex}} xyy \succ_{\mathrm{dlex}} yz \succ_{\mathrm{dlex}} x.$$

$$\underbrace{\begin{array}{cc} \begin{array}{cccc} xxyz & xyy & yz & x \end{array} & \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} & \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \end{array}}_{=\mathcal{M}_F} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Polynomials $\longleftrightarrow$ matrices

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$$\mathcal{F}_{\mathrm{RRef}(\mathcal{M}_F)} = \{xxyz + 5x, xyy - 2x, yz - 2x\}$$

First idea

Given $G = \{g_1, \ldots, g_k\} \subseteq K\langle X\rangle$ and $F = \{f_1, \ldots, f_m\} \subseteq K\langle X\rangle$ we want to reduce all $f \in F$ by G simultaneously.

First idea

Given $G = \{g_1, \ldots, g_k\} \subseteq K\langle X\rangle$ and $F = \{f_1, \ldots, f_m\} \subseteq K\langle X\rangle$ we want to reduce all $f \in F$ by G simultaneously.

$$\mathcal{M}_{G\cup F} = \begin{array}{c} \left(\begin{array}{cccc} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ \hline * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \end{array}\right) \end{array} \begin{array}{c} g_1 \\ \vdots \\ g_k \\ f_1 \\ \vdots \\ f_m \end{array}$$

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Symbolic preprocessing

Input: a finite set $F \subseteq K\langle X \rangle$ and $G \subseteq K\langle X \rangle$

Output: $G' \subseteq \{agb \mid a, b \in \langle X \rangle,\ g \in G\}$

$G' \leftarrow \emptyset$

$T \leftarrow \operatorname{supp}(F) \setminus \operatorname{lm}(F)$

$done \leftarrow \operatorname{lm}(F)$

while $T \neq \emptyset$ **do**

  select $t \in T$

  $T \leftarrow T \setminus \{t\}$

  $done \leftarrow done \cup \{t\}$

  if there exist $g \in G$, $a, b \in \langle X \rangle$ s.t. $t = a \operatorname{lm}(g) b$ **then**

    $G' \leftarrow G' \cup \{agb\}$

    $T \leftarrow T \cup (\operatorname{supp}(agb) \setminus done)$

return G'

Symbolic preprocessing

Input: a finite set $F \subseteq K[X]$ and $G \subseteq K[X]$

Output: $G' \subseteq \{tg \mid t \in [X], g \in G\}$

$G' \leftarrow \emptyset$
$T \leftarrow \mathrm{supp}(F) \setminus \mathrm{lm}(F)$
$\mathit{done} \leftarrow \mathrm{lm}(F)$
while $T \neq \emptyset$ **do**
  select $t \in T$
  $T \leftarrow T \setminus \{t\}$
  $\mathit{done} \leftarrow \mathit{done} \cup \{t\}$
  if there exist $g \in G$, $t' \in [X]$ s.t. $t = t' \,\mathrm{lm}(g)$ **then**
    $G' \leftarrow G' \cup \{t'g\}$
    $T \leftarrow T \cup (\mathrm{supp}(t'g) \setminus \mathit{done})$
return G'

Symbolic preprocessing

Let $(ABC - f', ABC - g')$ be the critical pair of an ambiguity of G.

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Example

Let $F = \{f_1, f_2, f_3, f_4\} \subseteq \mathbb{Q}\langle x, y\rangle$, with

$$f_1 = yxyx, \qquad f_2 = yxyx + yxx,$$
$$f_3 = yxy, \qquad f_4 = yxy + xy.$$

Let $G = \{g_1, g_2, g_3\} \subseteq \mathbb{Q}\langle x, y\rangle$, with

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Reduction

Input: a finite set $F \subseteq K\langle X\rangle$ and $G \subseteq K\langle X\rangle$

Output: $\tilde{F} \subseteq K\langle X\rangle \setminus \{0\}$

1: $G' \leftarrow \text{SymbolicPreprocessing}(F, G)$

2: $F' \leftarrow F \cup G'$

3: $\tilde{F} \leftarrow \{f \in \mathcal{F}_{\mathrm{RRef}(\mathcal{M}_{F'})} \mid f \neq 0 \text{ and } \mathrm{lm}(f) \notin \mathrm{lm}(F')\}$

4: **return** $\tilde{F}$

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4: **return** $\tilde{F}$

Main result

Theorem

Let $G \subseteq K\langle X\rangle$ and let C be a finite subset of critical pairs of ambiguities of G. Furthermore, let $F = \bigcup_{(\tilde{f},\tilde{g})\in C}\{\tilde{f},\tilde{g}\}$ be the set of all polynomials appearing in the critical pairs of C and let $\tilde{F} = \mathtt{Reduction}(F, G)$. Then,

$$\mathrm{spol}(a) \xrightarrow{*}_{G\cup\tilde{F}} 0,$$

for all ambiguities a of G such that the critical pair of a is in C.

F4 algorithm

Input: a finite set $\{f_1, \dots, f_m\} \subseteq K\langle X\rangle$

Output if algorithm terminates: $G \subseteq K\langle X\rangle$ GB of $(f_1, \dots, f_m)$

```
G ← {f_1, ..., f_m} \ {0}
critPairs ← critical pairs of all ambiguities of G
while critPairs ≠ ∅ do
    while critPairs ≠ ∅ do
        select C ⊆ critPairs
        critPairs ← critPairs \ C
        F ← ⋃_{(f̃,g̃)∈C} {f̃, g̃}
        F̃ ← Reduction(F, G)
        G ← G ∪ F̃
    critPairs ← critical pairs of all new ambiguities of G
return G
```

F4 algorithm

Input: a finite set $\{f_1, \ldots, f_m\} \subseteq K[X]$

Output: $G \subseteq K[X]$ GB of $(f_1, \ldots, f_m)$

$G \leftarrow \{f_1, \ldots, f_m\} \setminus \{0\}$

$\mathrm{critPairs} \leftarrow$ all critical pairs of G

while $\mathrm{critPairs} \neq \emptyset$ **do**

- **while** $\mathrm{critPairs} \neq \emptyset$ **do**
  - select $C \subseteq \mathrm{critPairs}$
  - $\mathrm{critPairs} \leftarrow \mathrm{critPairs} \setminus C$
  - $F \leftarrow \bigcup_{(\tilde{f}, \tilde{g}) \in C} \{\tilde{f}, \tilde{g}\}$
  - $\tilde{F} \leftarrow \mathrm{Reduction}(F, G)$
  - $G \leftarrow G \cup \tilde{F}$
- $\mathrm{critPairs} \leftarrow$ all new critical pairs of G

return G

Example

Let

$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy \qquad \in \mathbb{Q}\langle x, y \rangle.$$

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$$G = \{f_1, f_2, f_3\}$$

$$\begin{aligned} a_{12} &= (yx, 1, x, f_1, f_2), & a_{31} &= (yxy, 1, y, f_3, f_1), \\ a'_{31} &= (yxyx, yx, x, f_3, f_1), & a_{32} &= (yxy, yx, 1, f_3, f_2), \\ a'_{32} &= (yxy, 1, xy, f_3, f_2), & a_{33} &= (yxyxy, yx, xy, f_3, f_3) \end{aligned}$$

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$$F = \{yxyx, yxyx + yxx, yxy, yxy + xy\}$$

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$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy \qquad \in \mathbb{Q}\langle x, y\rangle.$$

$$G = \{f_1, f_2, f_3\}$$

$$\begin{aligned} a_{12} &= (yx, 1, x, f_1, f_2), & a_{31} &= (yxy, 1, y, f_3, f_1), \\ a'_{31} &= (yxyx, yx, x, f_3, f_1), & a_{32} &= (yxy, yx, 1, f_3, f_2), \\ a'_{32} &= (yxy, 1, xy, f_3, f_2), & a_{33} &= (yxyxy, yx, xy, f_3, f_3) \end{aligned}$$

$$F = \{yxyx, yxyx + yxx, yxy, yxy + xy\}$$

$$\xrightarrow{\text{SymPre}} F' = F \cup \{f_1 x, x f_2\}$$

Example

$$\underbrace{\begin{array}{c} \begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}}_{=\mathcal{M}_{\mathrm{F}'}}$$

Example

$$\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \end{array}}_{} $$

$$\underbrace{\begin{array}{c} \begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}}_{=\mathcal{M}_{\mathrm{F}'}} \xrightarrow{\mathrm{RRef}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \end{array}}{} $$

$$\underbrace{\begin{array}{c} \begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}}_{=\mathcal{M}_{\mathrm{F}'}} \overset{\mathrm{RRef}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \end{array}}_{=\mathcal{M}_{F'}} \overset{\text{RRef}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \end{array} \atop \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}}_{=\mathcal{M}_{F'}} \overset{\text{RRef}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \tilde{F} = \{xx, x\}$$

Example

$$
\underbrace{\begin{array}{cccccc} yxyx & yxy & yxx & xy & xx & x \\ \end{array}
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}}_{=\mathcal{M}_{F'}}
\xrightarrow{\text{RRef}}
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$\Longrightarrow \quad \tilde{F} = \{xx, x\}$$

$$G = \{f_1, f_2, f_3\} \cup \{xx, x\}$$

Example

$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

Example

$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

$$a_{14} = (yxx, y, x, f_1, xx), \qquad a_{15} = (yx, y, 1, f_1, x),$$

$$a_{35} = (yxy, y, y, f_3, x), \qquad a_{44} = (xxx, x, x, xx, xx),$$

$$a_{45} = (xx, x, 1, xx, x), \qquad a'_{45} = (xx, 1, x, x, xx).$$

Example

$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

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$$a_{14} = (yxx, y, x, f_1, xx), \qquad a_{15} = (yx, y, 1, f_1, x),$$

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$$F = \{yxx + xx, yxx, yx + x, yx\}$$

Example

$$f_1 = yx + x, \qquad f_2 = y + 1, \qquad f_3 = yxy$$

$$G = \{f_1, f_2, f_3, xx, x\}$$

$$a_{14} = (yxx, y, x, f_1, xx), \qquad a_{15} = (yx, y, 1, f_1, x),$$

$$a_{35} = (yxy, y, y, f_3, x), \qquad a_{44} = (xxx, x, x, xx, xx),$$

$$a_{45} = (xx, x, 1, xx, x), \qquad a'_{45} = (xx, 1, x, x, xx).$$

$$F = \{yxx + xx, yxx, yx + x, yx\}$$

$$\overset{\mathsf{SymPre}}{\Longrightarrow} F' = F \cup \{xx, x\}$$

Example

$$\underbrace{\begin{array}{cccc} yxx & yx & xx & x \\ \multicolumn{4}{c}{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \end{array}}_{=\mathcal{M}_{\mathrm{F}'}}$$

Example

$$\underbrace{\begin{array}{cccc} yxx & yx & xx & x \\ \end{array}}_{} $$

$$\begin{array}{c} \begin{array}{cccc} yxx & yx & xx & x \end{array} \\ \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=\mathcal{M}_{\mathrm{F}'}} \end{array} \overset{\mathrm{RRef}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$\underbrace{\begin{array}{cccc} yxx & yx & xx & x \\ \left(\begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right. & & & \left.\right) \end{array}}_{=\mathcal{M}_{\mathrm{F}'}} \xrightarrow{\mathrm{RRef}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$
\underbrace{\begin{array}{cccc}
yxx & yx & xx & x \\
\end{array}}_{}
$$

$$
\begin{array}{c}
\begin{array}{cccc} yxx & yx & xx & x \end{array} \\
\underbrace{\begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}}_{=\mathcal{M}_{\mathrm{F}'}}
\end{array}
\overset{\mathrm{RRef}}{\longrightarrow}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Example

$$\begin{array}{c} \begin{array}{cccc} yxx & yx & xx & x \end{array} \\ \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=\mathcal{M}_{F'}} \end{array} \xrightarrow{\mathrm{RRef}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies \tilde{F} = \emptyset$$

Example

$$\underbrace{\begin{array}{cccc} yxx & yx & xx & x \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}}_{=\mathcal{M}_{F'}} \overset{\text{RRef}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies \tilde{F} = \emptyset$$

$G = \{f_1, f_2, f_3, xx, x\}$ is a GB of (f_1, f_2, f_3)

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